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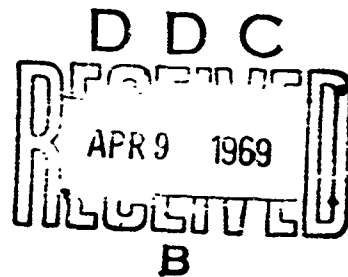### COLLEGE STATION, TEXAS

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DYNAMICS OF DEEP-SEA MOORING LINES

Robert O. Reid

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A&M Project 204
Reference 68-11F

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Texas A&M University

College of Geosciences
Department of Oceanography

College Station, Texas

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DYNAMICS OF DEEP-SEA MOO. ·· LINES

by

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PREFACE

The following report is the last of a series of five reports dealing with research related to the problem of deep-sea mooring lines under the stated contract. These studies were initiated in January, 1959 with Dr. Basil W. Wilson as principal investigator. A large part of the efforts, as summarized in the first four reports, were related to the problem of the equilibrium configuration. The present study, which was been completed under the support of the Texas A & M Research Foundation in fulfillment of the original contract obligations, deals with the dynamics of deep-sea mooring lines. The primary emphasis is on methodology in the solution of this problem. Although some application is made, an extensive parametric study is well beyond the scope of the present study.

Several publications based upon prior studies under this contract have appeared in the literature and are cited in the list of references.

The author wishes to express his appreciation for the patience of the original sponsor. Hs is particularly grateful for the preserverance of his secretary, Mrs. Florence G. McCully in coping with the lengthy set of equations endlessly interwoven into the fabric of the presentation. It is hoped that the reader is equally presevering.

R. O. Reid
Professor of Oceanography

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July, 1968

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I.   INTRODUCTION

1.   Background and Scope

The problem considered in this report deals with
the motions of and tensions within a quasi-elastic mooring
line which is anchored at the sea floor in oceanic depths
while attached to a ship or buoy at or near the sea
surface and subject to the influence of time varying
currents. This is a natural extension of the many pre-
vious studies dealing with the equilibrium configuration
of an anchored cable in the presence of steady, coplanar
currents. The mooring motion problem has received much
less attention.

One of the more recent investigations of the
equilibrium problem is that of Wilson (1964, 1965)*
in which the effect of vertical shear of the current is
included (in the second of the cited papers). A com-
prehensive summary of previous studies related to the
equilibrium configuration problem is given by Wilson (1964)
and will not be repeated here. The results of such
investigations of the equilibrium configuration of
mooring lines serve as a useful base on which one might

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* A list of references is given in alphabetical
order by author at the end of this report.

superimpose a dynamic perturbation theory related to departures of the current from its mean state.

In the last decade there has been increased interest in the general mooring motion problem, both in connection with the mooring of vessels in the deep sea and with the use of surface and subsurface anchored buoys, which serve as platforms for oceanographic and/or meteorological measurements. Recent studies which are particularly germane to the mooring motion problem are those of Whicker (1958), Walton and Polacheck (1960), Polacheck, et al. (1963), Paquette and Henderson (1965), and Fofonoff (1965, 1966). Although not directly related to the mooring problem, the paper of Zajac (1957) on the dynamics of laying and recovery of submarine cable should not be overlooked.

An important aspect of the problem of the transient mooring line, which does not enter directly into considerations of the equilibrium configuration, is that of the elastic and anelastic properties of the mooring line. Although the qualitative features of the stress-strain processes in stranded cables and ropes are well known, the available information is rather deficient in a quantitative sense. Moreover an adequate theory of the time-dependent properties of a rope or stranded cable

is lacking. Such a theory should be able to allow for such phenomena as creep under nearly constant load, time-dependent relaxation of elastic strain under no load and hysteresis under conditions of cyclic loading.

In a recent paper by Wilson (1967) a summary of some of the important properties of nylon and coir ropes and of stranded steel cables is presented. This includes information in regard to the weight per unit length in water, the _dynamic_ modulus of elasticity (which is a function of the mean load) and the permanent or anelastic strain (which depends upon the maximum tension that the material has previously experienced). Quantitative measurements in regard to hysteresis and creep, as well as dynamic modulus of elasticity for nylon rope, under cyclic loading at various mean tensions, have been carried out recently by Paquette and Henderson (1965). However, the influence of hysteresis was not allowed for in their transient state analysis of a moored buoy. An attempt is made in the present study to include the hysteresis phenomenon by adopting a time-dependent model for the physical properties of the mooring line, consistent with the above information. This is one of the new aspects of the present theory.

The dynamical model adopted in this study allows for three degrees of freedom of displacement of each material point of the mooring line in response to currents, which are not necessarily in the same direction at all depths, and/or in response to three-dimensional motions of the floatation unit (ship or buoy) to which the mooring line is attached. The mooring line is regarded as completely flexible but extendable (both in the elastic and anelastic sense). Hysteresis and relaxational phenomena related to the time-dependent stress-strain processes are admitted via a generalized, Maxwell type, visco-elastic model of the physical behavior of the mooring line. This model incorporates a "memory effect", reflecting the influence of the prior history of the stress-strain process.

The damping of longitudinal transient oscillations of the mooring line is related primarily to the hysteresis properties of the line. On the other hand, the damping of transverse oscillations is primarily related to the mechanical "radiation" of energy to the fluid through the action of hydrodynamic form drag, which is dependent upon the motion of the mooring line relative to the fluid. These two effects are considered to be the dominant mechanisms for damping in the present theory. Specifically, the tangential skin drag on the cable (Reid and Wilson, 1963) is regarded as negligible compared with the normal

form drag. Moreover, as in most other studies of cable dynamics, thermal effects are not considered, nor is the influence of hydrostatic pressure on the properties of the cable (this being unknown at the present time).

2. Order of Presentation

Within the framework of the above restrictions, a "general" theory of the three-dimensional mooring line dynamics is set forth. The presentation starts with a formulation of the equations of motion of the mooring line in vector form and a discussion of the hydrodynamical fluid forces on the line. This is followed by a discussion of the time-dependent stress-strain properties of ropes and stranded cables and presentation of an anelastic model. The implications of this model in respect to the dynamic modulus of elasticity and hysteresis are then compared with the data of Paquette and Henderson (1965) and Wilson (1967).

A general consideration of the energy conservation relation for the system is presented. The damping effects associated with both form drag and internal hysteresis are displayed explicitly along with the energy supply terms due to work on the mooring system at its boundaries.

In the section which follows, the component equations
of motion of the mooring line are given explicitly in a
natural coordinate system similar to that suggested by
Zajac (1957), which is based on the local orientation of
the mooring line. This system has the advantage of
facilitating the evaluation of the normal drag force on
the mooring line. Kinematical compatibility conditions
are derived which relate the component velocities of the
mooring line to other dependent parameters characterizing
the cable orientation and strain. The system of equations
are of quasi-linear, hyperbolic type and accordingly
can be recast in the so-called characteristic form
(Freeman, 1951). The latter form has the advantage
that a stable numerical solution of the mooring line
configuration at any time t, starting with an arbitrary
initial three-dimensional configuration is greatly
facilitated. The characteristic form of the basic
relations also clarifies the roles played by the trans-
verse and longitudinal modes of motion of the mooring
line and hence adds to our understanding of the mooring
line dynamics. The numerical method of solution via
the method of characteristics is discussed.

For certain problems the linearized equations of motion of the mooring line are of advantage. These equations are set forth in a subsequent section and some examples of their use for small perturbations of the mooring line from a straight equilibrium state are discussed.

The use of the perturbation approximation is then extended to the more general case where the equilibrium configuration of the mooring line is a curved line in the vertical plane. In this case the longitudinal and transverse modes of oscillation of the mooring line relative to the equilibrium state are coupled due to the curvature. The linearized perturbation approximation is useful in respect to considerations of the stochastic aspects of the response of the mooring line to irratic motion of the floatation unit to which it is attached. An application of the method is made for the case where the motion of the floatation unit is prescribed in terms of a stochastically stationary time sequence with stipulated variance spectrum.

II. EQUATION OF MOTION OF MOORING LINES

1. Assumptions

The primary assumptions in the present analysis concerning the dynamics of a mooring line are as follows:

a. The mooring line is treated as perfectly flexible, i.e., it is supposed that the line cannot support shearing stresses normal to the line;

b. Torsional strains in the line are not considered, it being assumed that there is no effect of such strains on the longitudinal or transverse displacements of the line;

c. Longitudinal skin drag on the line is neglected compared with normal drag associated with relative fluid motion past the line;

d. It is considered that the mooring line is fixed at its lower end; i.e., it does not drag anchor;

e. It is supposed that the longitudinal
    strain in the line is related to the
    axial tension and its time history only,
    thermal strains and exchange of heat with
    the environmental fluid being ignored;

f. It is assumed that radial strain of the
    line is related directly to the longitudinal
    strain and is independent of the pressure
    exerted by the surrounding fluid;

g. Finally the Coriolis force on the mooring
    line is considered altogether negligible
    compared with other forces.

In the general analysis, no restriction is imposed
on the nature of the current acting on the mooring line.
Accordingly allowance is made for three degrees of
freedom of displacement of any material point of the
line. In general the mooring line is a three-dimensional
curve at any instant. However, certain special cases
will be considered in which the equilibrium configuration
lies in a vertical plane.

Assumptions (e) and (f) are imposed largely in view
of the limited state of knowledge in regard to the physical
and thermal properties of typical mooring line materials.

The radial strain is needed only in respect to the
evaluation of the buoyancy and normal drag force on
the line, which depends on its effective diameter among
other factors.

2. <u>Material Coordinate and the Vector Equation of Motion</u>

Consider a mooring line which is securely anchored
at point O on the sea bed (Figure 1). Let P be any
material point on the axis of the mooring line.* Let $\hat{r}$
denote the position vector of P with respect to anchor
point O as a reference. Clearly the mass, M, of line
between O and P is conserved regardless of the
dynamic state of the line. One could accordingly adopt
M as a logical material coordinate of the point P.
However, it is more customary to employ an arc length
to identify the point P. In the present analysis we will
let s denote the arc length between the two material
points O and P of the mooring line in its <u>original</u>,
<u>completely</u> <u>relaxed</u> state, prior to any permanent strain.
We also define m as the mass per unit length at s
in the <u>same state</u>, such that

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* A list of symbols appears at the end of this
report for convenience to the reader.
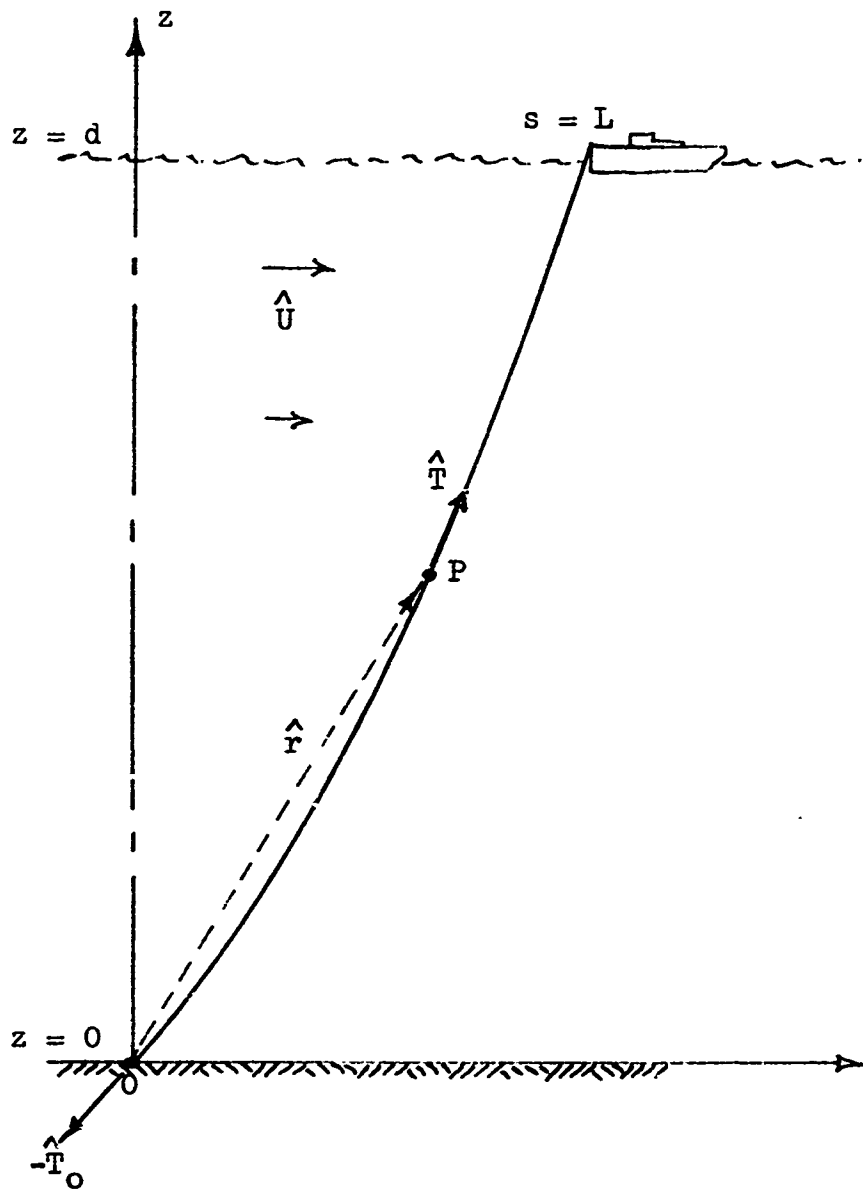
Fig. 1   Schematic of mooring line.

$$M = \int_0^s m(s)\ ds, \qquad\qquad (1)$$

in which m (for a tapered line) may in general depend on s. In this sense s is merely an alias of M.

Note that s is <u>not</u> to be confused with the actual arc length, σ, of the line between O and P in its general strained state. In general $\sigma > s$. Moreover the mass per unit actual length under strained conditions will generally be less than m, and is equal to $m\ ds/d\sigma$.

All dependent variables such as $\hat{r}$, σ, etc. will be regarded as functions of the independent variables s and t. To avoid confusion with σ we will hereafter refer to s as the <u>material coordinate</u>.

The vector velocity of any material point of the line will be denoted by $\hat{V}$ and is related to $\hat{r}(s,t)$ by

$$\hat{V} = \partial\hat{r}/\ \partial t. \qquad\qquad (2)$$

The translational momentum of the line between O and P is accordingly given by

$$\int_0^s m\ \hat{V}\ ds.$$

Let $\hat{T}$ denote the vector tension at P and $\hat{T}_o$ the vector tension at O, both taken in the sense of increasing s along the line. These tensile loads are, in view of assumption (a), tangent to the axis of the line at the points in question. Let $\hat{k}$ denote the vertical unit vector (upwards) and g the acceleration due to gravity such that in the absence of fluid the line between O and P has the weight $-M g \hat{k}$ or

$$-\int_0^s m g \hat{k} \, ds.$$

Finally let $\hat{F}$ denote the residual surface force acting on the line between O and P due to the action of the surrounding fluid.

Application of Newton's law, bearing in mind assumptions (a) and (g), then leads to the relation

$$\partial/\partial t \int_0^s m \hat{V} \, ds = \hat{T} - \hat{T}_o - \int_0^s m g \hat{k} \, ds + \hat{F}. \qquad (3)$$

Now m is independent of t and $\hat{T}_o \equiv \hat{T}(o,t)$, so it follows from (3) by differentiating with respect to s that

$$m \, \partial\hat{V}/\partial t = \partial\hat{T}/\partial s - m g \hat{k} + \hat{f}, \qquad (4)$$

in which T has the direction of the vector $\partial\hat{r}/\partial s$ and $\hat{f} \equiv \partial\hat{F}/\partial s$.

3. Hydrodynamic Forces Due to the Fluid

In the absence of any motion of the fluid or of the mooring line, the force $\hat{F}$ reduces to the static Archemedian buoyance force, $\hat{B}$, given by

$$\hat{B} = g\hat{k} \int_{0}^{\sigma} \rho A \, d\sigma \qquad (5)$$

. which ρ is the fluid density (mass per unit volume) and A is the cross-sectional area of the material of the line at s under strained conditions. The buoyancy per unit material coordinate is accordingly given by

$$\hat{f}_{b} \equiv \partial\hat{B}/\partial s = \rho \, g \, \hat{k} \, A \, \partial\sigma/\partial s \, . \qquad (6)$$

For simplicity it will be supposed that ρ is a constant. However A and $\partial\sigma/\partial s$ can in general vary with s or t.

In the presence of motion of the fluid relative to the mooring line, an additional dynamic force can occur. This is associated with anomalies of pressure distribution around the surface of the immersed line. Such pressure

anomalies can be created by acceleration of the fluid and/or by acceleration of the line through the fluid. In addition, at moderate to large Reynolds number, a wake phenomenon can exist on the lee side of the line, which leads to a form drag effect above and beyond the influence of acceleration. Also it is well known that periodic vortex shedding, which occurs at moderate Reynolds number, can lead to an alternating lateral force on the line (normal to the relative current) at a frequency dependent upon the the Reynolds number (Strouhal frequency).

The drag force, $\hat{f}_d$, per unit material coordinate is presumed to be of the form

$$\hat{f}_d = (1/2)\rho \; c_d \; D \; |\hat{U}_n - \hat{V}_n| \; (\hat{U}_n - \hat{V}_n) \; \partial\sigma/\partial s, \qquad (7)$$

where D is the effective diameter of the line at position s and time t, c_d is the drag coefficient which is a function of the Reynolds number associated with the relative speed, $\hat{V}_n$ is the component of $\hat{V}$ normal to the mooring line at s,t and U_n is the component of the fluid velocity $\hat{U}$ normal to the mooring line at s, t*. The above relation gives a vector force

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* Actually $\hat{U}$ is the fluid velocity that would exist in the absence of the mooring line.

normal to the line at s, t, the tangential skin drag
being neglected. Values of c_d for normal drag on
stranded cable, as a function of the Reynolds number,
are summarized by Wilson (1964). The tangential drag
coefficient is of the order of only one per cent of that
for normal drag for usual ranges of Reynolds number
(see Wilson, 1964 or Reid and Wilson, 1963). Accordingly
the omission of tangential skin drag is not considered
serious. The vector $\hat{U}_n$ can be expressed in the form

$$\hat{U}_n = \hat{U} - \hat{\tau}\, U_\tau \tag{8}$$

where $\hat{\tau}$ is a unit vector tangent to the axis of the
mooring line at s, t and $U_\tau = \hat{\tau} \cdot \hat{U}$. A similar
relation of course holds for $\hat{V}_n$. The unit vector $\hat{\tau}$
is related to $\hat{r}$ by the relation

$$\hat{\tau} = \frac{\partial \hat{r}}{\partial \sigma} = \frac{\partial \hat{r}}{\partial s} \Big/ \frac{\partial \sigma}{\partial s} \; . \tag{9}$$

The effect of vortex shedding can be simulated by
allowing an oscillatory component of $\hat{f}_d$ normal to the
direction of $\hat{f}_d$ and the local vector $\hat{\tau}$. For the
evaluation of the Strouhal frequency at given Reynolds
number, see for example Schlichting (1960). The amplitude

of the resulting transverse, alternating force is
presumably of the order of magnitude of ten per cent
of the local mean value of $\hat{f}_d$. If we let η denote the
ratio of the amplitude of lateral force to the drag force
and let ω_s be the Strouhal frequency, then the alter-
nating lateral thrust per unit length due to vortex
shedding can be approximated by

$$\hat{f}_s = \eta \, \hat{\tau} \times \hat{f}_d \cos (\omega_s t + \delta) \, , \qquad (10)$$

where δ is an arbitrary phase angle.

To the above drag force, which can exist under
steady relative motion, we must allow for the possibility
of an added inertial force. Consider for the time being
that the fluid motion $\hat{U}$ (away from the influence of the
mooring line) is steady, but that the mooring line is
accelerated in a direction normal to the line like a
rigid cylinder. In this case a portion of the fluid
in the neighborhood of the line must also be accelerated,
thus leading to an effective retarding force on the line.
The magnitude of this force per unit length of line can
be expressed as $- m_a a_n$ where a_n is the acceleration
normal to the line and m_a is the effective added mass
of fluid being accelerated at the rate a_n. If the line

experiences an acceleration tangential to the line and if
viscous effects are ignored then there will be essentially
no acceleration of the fluid if end effects are also
ignored (very long line). Thus we should expect the
inertial force to be highly directional, like the drag
force.

For a general acceleration of the line but with $\hat{U}$
steady, we propose the following expression for the
inertial force per unit length due to the presence of
the fluid

$$\hat{f}_i' = - m_a \, (\partial \hat{V}/\partial t - \hat{\tau} \, \partial V_\tau/\partial t) \, , \qquad (11)$$

where $V_\tau \equiv \hat{V} \cdot \hat{\tau}$, this being the tangential component
of the mooring line velocity at s, t. The above relation
is not the only possible choice for $\hat{f}_i'$. Other logical
choices are

$$- m_a \, [\, \partial \hat{V}/\partial t - \hat{\tau} \, (\partial \hat{V}/\partial t) \cdot \hat{\tau} \,]$$

and

$$- m_a \, \partial/\partial t \, \, (\hat{V} - \hat{\tau} \, V_\tau) \, .$$

If $\hat{\tau}$ were independent of time then each of the three
possible forms yield identical values of $\hat{f}_i'$. Thus

the differences result only from terms involving $\partial\hat{\tau}/\partial t$, which represents a turning rate associated with the local orientation of the axis of the mooring line. Of the above three possible forms for $\hat{f}'_i$ (for steady $\hat{U}$), only that given by (11) leads to a physically acceptable energy equation for the system as we will see in a later section. The choice of relation (11) is justified primarily on this basis.

In the more general case where the velocity of the fluid, $\hat{U}$ (in the absence of the mooring line) is not steady, then the acceleration term $\partial\hat{U}/\partial t$ will contribute to the inertial force. Its effect, above and beyond that of $\partial\hat{V}/\partial t$, is analysed as follows. In the absence of the mooring line, the fluid occupying the position of the mooring line would experience a force $\rho A\ \partial\sigma/\partial s$ $\partial\hat{U}/\partial t$, per unit s, where $A\ \partial\sigma/\partial s$ represents the volume per unit s. This force exists by virtue of the pressure gradient required to produce the acceleration $\partial\hat{U}/\partial t$. In the presence of the mooring line, a disruption of the flow occurs in its vicinity and this introduces an added inertial force similar to that caused by accelerating the mooring line through the fluid. The added inertial force is taken as

$$m_a \ (\partial \hat{U}/\partial t - \hat{\tau} \ \partial U_\tau/\partial t) \ ,$$

by analogy with (11). Thus in the absence of any acceleration of the mooring line the inertial force per unit of s is considered to be given by

$$\hat{f}''_i = \rho \ A \ \partial \sigma/\partial s \ \ \partial \hat{U}/\partial t$$

$$+ m_a \ (\partial \hat{U}/\partial t - \hat{\tau} \ \partial U_\tau/ \ \partial t) \ . \tag{12}$$

The total inertial force per unit of s exerted by the fluid, in the presence of acceleration of the fluid as well as acceleration of the mooring line, is accordingly taken as the sum of the right hand sides of (11) and (12).

The added mass coefficient m_a per unit s can be expressed in the form

$$m_a = \rho \ A \ c_a \ \partial \sigma/\partial s \tag{13}$$

where c_a is a non-dimensional added mass coefficient. The theoretical value of c_a for pure potential flow around a circular cylinder according to Lamb (1945)

is exactly unit. This value applies for very small
Reynolds number. For large Reynold number, the value
of c_a is generally somewhat less than unity.

4. Alternative Form of the Equation of Motion

The total force per unit material coordinate due
to the fluid is given by

$$\hat{f} = \hat{f}_b + \hat{f}_d + \hat{f}_s + \hat{f}'_i + \hat{f}''_i \quad . \tag{14}$$

Making use of (6), (7), (10), (11), (12), and (14) in
(4) and combining terms involving the acceleration
of the mooring line yields

$$(\bar{m} + \bar{m}_a)\, \partial\hat{v}/\partial t - m_a\, \hat{\tau}\, \partial v_\tau/\partial t = \partial\hat{T}/\partial s - w\hat{k} + \hat{R}, \tag{15}$$

where w is the net weight per unit s given by

$$w = (m - m_d)\, g \tag{16}$$

in which m_d is the displaced mass of fluid per unit s

$$m_d = \rho\, A\, \partial\sigma/\partial s. \tag{17}$$

The residual fluid force $\hat{R}$ is given by

$$\hat{R} = \frac{\rho}{2} c_d \, D \frac{\partial \sigma}{\partial s} |\hat{U}_n - \hat{V}_n| \, [1 + \eta \cos (\omega_s t + \delta) \, \hat{\tau}x] \, (\hat{U}_n - \hat{V}_n)$$

$$+ (m_d + m_a) \frac{\partial \hat{U}}{\partial t} - m_a \, \hat{\tau} \, \frac{\partial U_\tau}{\partial t} \quad . \tag{18}$$

It will be recalled that

$$m_a = c_a \, m_d \tag{19}$$

and

$$\hat{U}_n = \hat{U} - \hat{\tau} \, U_\tau$$

$$\hat{V}_n = \hat{V} - \hat{\tau} \, V_\tau \tag{20}$$

where $U_\tau = \hat{U} \cdot \hat{\tau}$ and $V_\tau = \hat{V} \cdot \hat{\tau}$. Moreover, $\hat{V}, \hat{\tau}$ and $\partial \sigma / \partial s$ are related to $\hat{r}$ as follows

$$\hat{V} = \partial \hat{r} / \partial t \tag{21}$$

$$\hat{\tau} = \frac{\partial \hat{r}}{\partial s} / \frac{\partial \sigma}{\partial s} \tag{22}$$

$$\frac{\partial \sigma}{\partial s} = |\frac{\partial \hat{r}}{\partial s}| . \tag{23}$$

The vector tensile force can be expressed as

$$\hat{T} = T\,\hat{\tau} \tag{24}$$

where T is the scalar tension at s, t and is related
to the longitudinal strain $(\partial\sigma/\partial s - 1)$ by an equation
of state appropriate to the particular mooring line
concerned. This will be considered in some detail in a
later section.

The dependent variables $\hat{r}$, $\hat{V}$, $\hat{\tau}$, $\hat{T}$ and σ are
sought as functions of s and t for given initial
and end conditions. The velocity $\hat{U}$ is understood to
be a prescribed function of $\hat{r}$ and t. Its dependence
on s is not given explicitly but is implied through
the dependence of $\hat{r}$ on s.

5. End Conditions - Vessel Motion

At the anchor point for each mooring line we require
simply

$$\hat{r}\,(o,\, t) = 0, \tag{25}$$

which implies in turn that $\hat{V}\,(o, t)$ also vanishes.

We will restrict our analysis to mooring lines
without any intermediate attachments between the anchor
and the surface vessel to which each is secured. Let
L denote the material coordinate at the point of

attachment of the given mooring line to the vessel (i.e.,

L is the original unstrained total length of the mooring

line). A possible simple end condition at $s = L$ is

to specify $\hat{V}$ (L, t) as a prescribed function of time.

This together with specified initial values of $\hat{r}$ and

$\hat{V}$ versus s is admissible from a mathematical point of

view. However, from a physical point of view, it does

not seem rational to specify the motion of the moored

vessel a priori. Clearly the action of the mooring line

plays a role in constraining the vessel's motion and

indeed can introduce natural frequencies of oscillation

in addition to those which are relevant to an unmoored

vessel. The unmoored vessel, of course, possesses

distinct natural frequencies for roll, pitch, and heave

only. The constraint imposed by the mooring line (or

lines) can lead to additional natural frequencies in

surge, sway, and yaw (O'Brien and Muga, 1963).

Thus it would be desirable to deduce the motion

of the moored vessel itself in terms of the hydrodynamic

forces acting on it, together with the constraining effect

of the mooring line. This implies that $\hat{r}$ (L, t) and

$\hat{T}$ (L, t) for the mooring line (or lines) are related

by an appropriate differential equation in which the

wave action and/or steady current acting on the vessel

should enter as the prescribed exciting force. The

approach employed here is a compromise in which the motion
of the vessel in the absence of the mooring line (the
potential motion) is prescribed rather than the exciting
forces. This of course requires the specification of six
scalar functions of time corresponding to the six degrees
of freedom of motion for a rigid vessel. Typical measure-
ments of roll, pitch, and heave for large vessels underway
in a seaway are given, for example by Gover (1955).
Measurements of all six motions for a large moored vessel
are presented by O'Brien and Muga (1963).

Approximate relations connecting the motion of a
moored vessel with the prescribed potential motion for
given mooring conditions are derived in Appendix A
based upon the linearized equations of motion of the type
employed by St. Denis and Pierson (1955), but allowing
for the constraint imposed by the mooring line (or lines).
Allowance is made for torque exerted by the mooring lines
as well as the constraining force since in general the
attachment of a given mooring line is not located at the
center of the mass of the vessel. The resulting equations
are valid for small translational and rotational displace-
ments of the vessel. These relations are summarized
below.

Let $\hat{r}_k$ denote the position vector of the point
of attachment on the vessel of mooring line k relative

to the anchor point of mooring line k. Let $\hat{p}_k$ denote
the separation vector from the vessel's center of mass (C)
to the attachment point of mooring line k (Fig. 2).
The vector $\hat{p}_k$ has constant magnitude but varies in
direction depending on the angular displacement of the
vessel. The mean values of the vectors $\hat{r}_k$ and $\hat{p}_k$
are denoted by $\hat{r}_k{}^o$ and $\hat{p}_k{}^o$ respectively. Now let
$\hat{\lambda}$ and $\hat{\alpha}$ denote the potential translational and
rotational displacements* of the vessel, in a given
seaway, if it were not moored and not underway. These
two vectors are regarded as prescribed functions of
time. For small rotational displacements, the position
vectors $\hat{r}_k$ at the attachment points can be approximated
by

$$\hat{r}_k = \hat{r}_k{}^o + \hat{\lambda} + \hat{\alpha} \times \hat{p}_k{}^o$$
$$+ \hat{R}' + \hat{\psi}' \times \hat{p}_k{}^o \tag{26}$$

where $\hat{R}'$ and $\hat{\psi}'$ represent anomalies of the translational
and rotational displacements of the vessel associated with
the constraint of the mooring lines. Thus $\hat{\lambda} + \hat{R}'$ is the
vector translational displacement of the center of mass

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* Referred to equilibrium state.
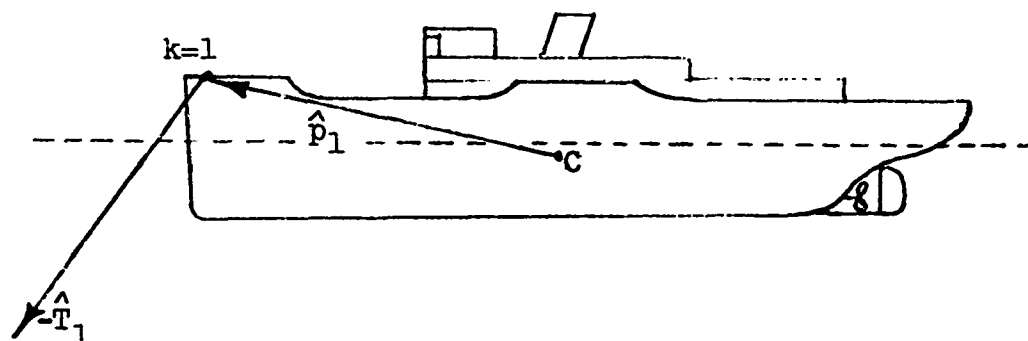
Fig. 2 Schematic illustrating terms employed in the upper end condition.

of the moored vessel relative to its equilibrium position.

Likewise $\hat{\alpha} + \hat{\psi}'$ is the vector rotational displacement

of the moored vessel relative to its equilibrium

orientation.

Let R_1', R_2', R_3' denote respectively the surge,

sway, and heave components of the vector $\hat{R}'$ and let

ψ_1', ψ_2', ψ_3' denote respectively the components of

$\hat{\psi}'$ in roll, pitch, and yaw. These components represent

a Cartesian system, the reference axes of which are taken

as the equilibrium orientation of the principal axes of

the vessel. As shown in Appendix A, the approximate

equations of motion governing the above quantities are

(in compact form) -

$$(d^2/dt^2 + 2\beta_j \, d/dt + \sigma_j^2) \, R_j' = -\sum_k T'_{kj}/M_j \qquad (27)$$

and

$$(d^2/dt^2 + 2\beta_j' \, d/dt + \sigma_j'^2)\psi_j' = -\sum_k J'_{kj}/I_j \qquad (28)$$

where $j = 1, 2$, or 3. The sums on the right are taken

over the total number of mooring lines. The terms

T'_{kj} represent the Cartesian components* of the vector

---

*In the same system as that for R_j' and ψ_j'.

$\hat{T}_k$' where

$$\hat{T}_k' \equiv \hat{T}_k - \hat{T}_k^o , \qquad\qquad (29)$$

$\hat{T}_k$ being the vector tensile force at the point of attachment on the vessel for line k (Fig. 2) and $\hat{T}_k^o$ is the mean value of $\hat{T}_k$ (or equilibrium value). Likewise the terms J'_{kj} are the Cartesian components, in the same system of the vector $\hat{J}_k'$ defined by

$$\hat{J}_k' \equiv \hat{J}_k - \hat{J}_k^o , \qquad\qquad (30)$$

where

$$\hat{J}_k \equiv [\hat{p}_k + (\hat{\psi}' + \hat{\alpha}) \times \hat{p}_k] \times \hat{T}_k \qquad\qquad (31)$$

and $\hat{J}_k^o$ is the mean value of $\hat{J}_k$. Physically $-\hat{J}_k$ represents the torque exerted by line k on the vessel.

The terms M_j are the effective masses of the vessel (including effective added mass of water) for motion in surge, sway, and heave (j = 1, 2, 3 respectively). The terms I_j are the effective moments of inertia about the principal axes for roll, pitch, and yaw (j = 1, 2, 3 respectively). The coefficients β_j and β_j' are

are damping rates (units of reciprocal time) for the six components of motion. Finally σ_j and σ_j' are the nominal natural frequencies (radians per unit time) for the unmoored vessel. Note that $\sigma_1 = \sigma_2 = \sigma_3' = 0$ (for surge, sway, and yaw).

Equations (27) and (28) imply that $\hat{R}'$ and $\hat{\psi}'$ are appropriate integrals of the dependent vector $\hat{T}_k$ and specified function $\hat{\alpha}$. Thus (26) implies a relation between the dependent variable $\hat{r}_k$ at the upper end of line k and the $\hat{T}_k$ for all lines. Moreover, there are as many relations (26) as there are lines, which implies that the system should be determinate. As a special case, if $\hat{R}'$ and $\hat{\psi}'$ are negligible compared with $\hat{\lambda}$ and $\hat{\alpha}$ respectively, then (26) reduces to the simple relation

$$\hat{r}_k = \hat{r}_k{}^o + \hat{\lambda} + \hat{\alpha} \times \hat{p}_k{}^o \qquad (26a)$$

in which $\hat{\lambda}$ and $\hat{\alpha}$ are prescribed.

Whether one chooses to specify $\hat{\lambda}$ and $\hat{\alpha}$ or $(\hat{\lambda} + \hat{R}')$ and $(\hat{\alpha} + \hat{\psi}')$ is perhaps optional; in either case the motion of the upper end of the mooring lines (in the case of multiple mooring) should be consistent with (26).

III. STRESS-STRAIN RELATIONS FOR
ROPES AND STRANDED CABLES

1. Known Properties

Some examples of load versus elongation for ropes
and stranded cables are shown in Figs. 3 and 4. In
Fig. 3, which is taken from Wilson (1967), typical
processes for steel wire cables (upper panels) and for
coir mooring rope (lower panels) are presented. Fig. 4,
which is taken from Paquette and Henderson (1965),
represents a test carried out for a one-half inch
diameter nylon rope of 55-inch original length. In the
latter test the tension was cycled over ranges of $\pm$ 180
lbs. relative to ten selected mean tensions. Such
tests demonstrate certain features common to both
stranded cables and ropes which are summarized below:

a. The stress-strain relation is definitely
process dependent; i.e., the relation between stress
and stain at time t is dependent upon the prior
stress-strain conditions;

b. The material suffers a permanent anelastic
strain which is dependent upon the maximum load
previously experienced;

c. For cyclic loading and unloading a hysteresis loop exists which implies that the stored energy is not entirely elastic;

d. Anelastic creep is evident for cyclic loading particularly for large mean loads; the creep tends to reach a limit after many cycles so as to form a closed hysteresis loop;

e. The amount of hysteresis energy loss per cycle definitely depends upon the range of cyclic load;

f. The material can experience a time dependent relaxation of strain (negative creep) upon sudden release of load;

g. Under cyclic loading, the apparent spring coefficient (slope of the principal axis of the hysteresis loop for small range of load) increases with increasing mean load (Fig. 4).

The latter effect can also be seen in terms of the curved axis of the large hysteresis loops shown in Fig. 3 (dashed curves in upper panels). The rope or cable acts like a stiffening spring as well as displaying significant hysteresis.

Conclusive measurements are lacking in regard to the influence of cycling rate on the apparent spring coefficient and hysteresis for a given mean load.
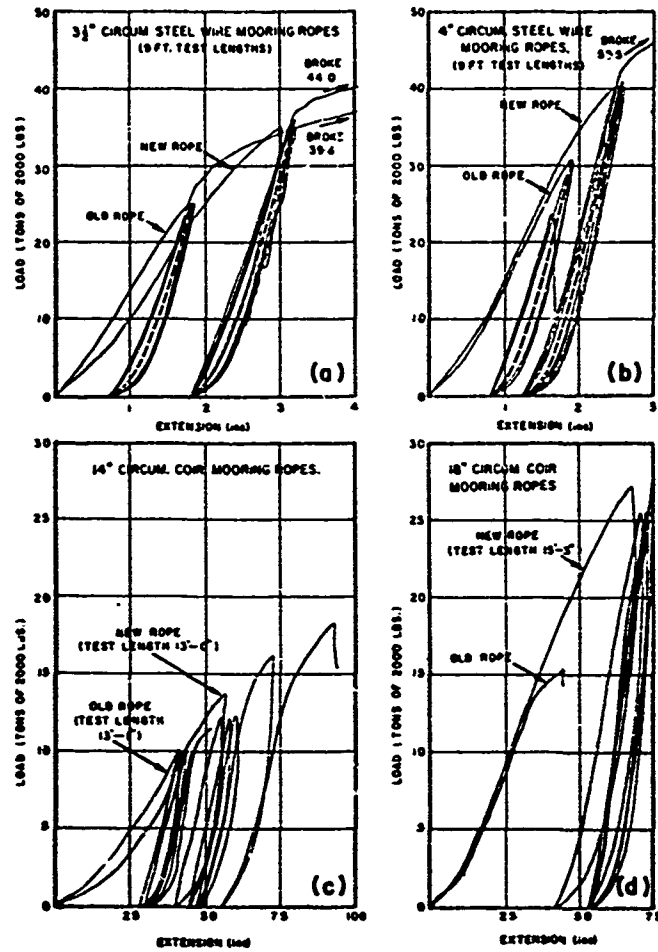
Fig. 3   Repeated load-elongation tests for mooring
          line materials: (a) and (b) steel-wire
          ropes; (c) and (d) coir fiber ropes (after
          Wilson, 1967).
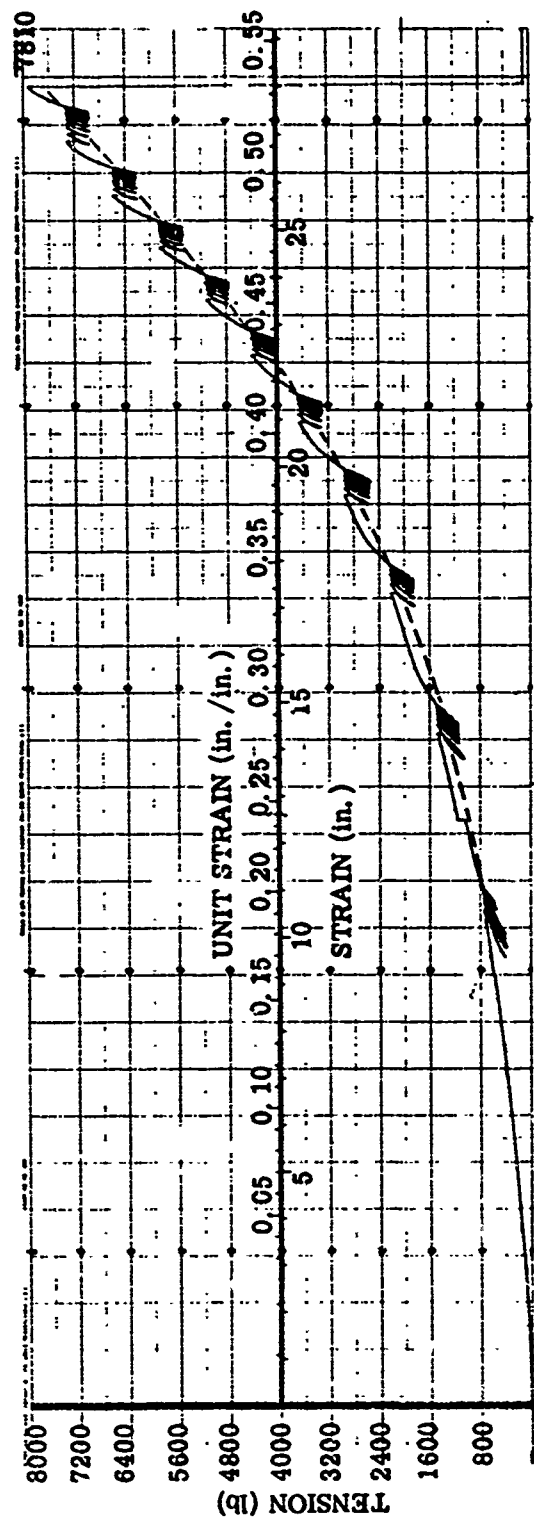
Fig. 4   Record of dynamic load-strain tests for one-half inch diameter
nylon rope (after Paquette and Henderson, 1965).

2. A Simple Maxwell Model

A possible approach for representing the stress-strain processes in a rope or cable is to adopt a simple three-parameter Maxwell model to simulate its visco-elastic properties (Freiberger, 1960, p. 579 and 774). Such a model is shown schematically in Fig. 5. In this model K_1 and K_0 represent spring coefficients with units of force per unit strain and N is a frictional coefficient with units of force per unit time rate of strain. Let ϵ be the overall strain for the system; this is the same as the strain in segment DE or in segment AC (Fig. 5). Let ϵ_1 and ξ be the partial strain in segments between AB and BC, respectively. Let T_1 be the common tension in the series segments AB and BC and let T_0 be the tension in segment DE. The total load, T, is given by $T = T_0 + T_1$; moreover $\epsilon = \epsilon_1 + \xi$. Finally we take $T_1 = K_1 \epsilon_1 = N \, d\xi/dt$ and $T_0 = K_0 \epsilon$. Eliminating T_0, T_1, ϵ_1 and ϵ_2 between the above five relations leads to the following constitutive equation for the system

$$T + \tau \, dT/dt = K_0 \epsilon + (K_0 + K_1) \, \tau \, d\epsilon/dt, \qquad (32)$$
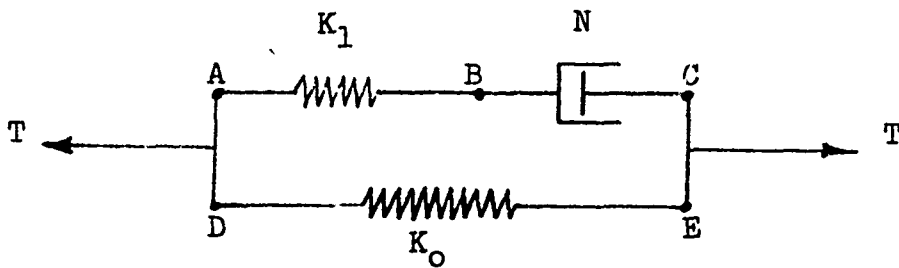
Fig. 5   Schematic of a simple visco-elastic
          Maxwell model (symbols are explained
          in the text).
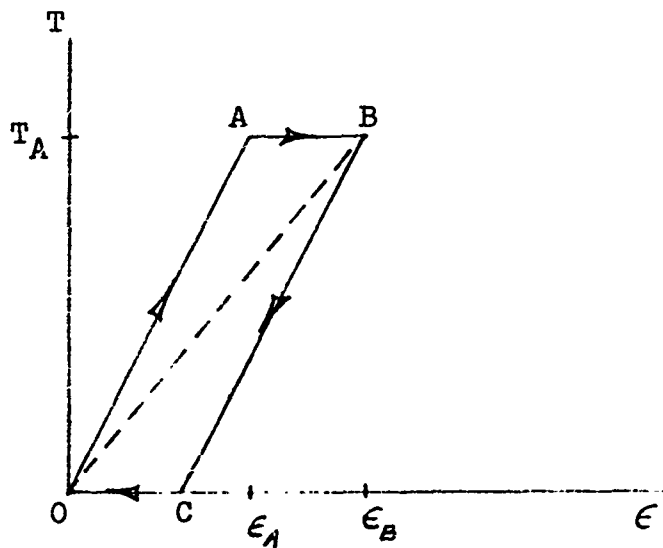
Fig. 6   Idealized load-strain process for the
          Maxwell model; the slope of lines OA and
          CB is $(K_0 + K_1)$; the slope of line OB
          is K_0.

where $\tau = N/K_1$. The latter parameter represents a
characteristic relaxation time for the system. In
the above relation it is presumed that the three
parameters K_0, K_1, and τ are constants. A relation
essentially equivalent to (32) was deduced by Meixner
(1965) based upon an entirely different approach in-
volving the thermodynamic non-equilibrium processes
in anelastic materials.

The above model implies that for very slow changes
(32) reduces to

$$T \doteq K_0\, \epsilon, \tag{33}$$

while for rapid changes

$$\Delta T \doteq (K_0 + K_1)\, \Delta\epsilon. \tag{34}$$

Thus the system is characterized by two limiting elastic
stress-strain relations which we will refer to as static
and dynamic.

Consider the following special time dependent
process illustrated in Fig. 6:

a. Starting at zero load and zero strain apply
a sudden load T_A resulting in elongation $\epsilon_A = T_A/(K_0 + K_1)$.

b. Under the sustained load T_A, (32) predicts
that ϵ will creep ultimately to the value $\epsilon_B = T_A/K_0$
after an elapsed time of many τ units.

c. Having reached point B now suppose the
load is suddenly removed; the strain immediately after
removal should be $\epsilon_C = \epsilon_B - \epsilon_A$ since curve BC is
parallel to curve OA.

d. Again at constant (zero) load, the system
is not in equilibrium and (32) predicts a relaxation of
the strain with nearly complete recovery after an elapsed
time of many τ units.

The above process is an idealized hysteresis loop;
in fact it gives the maximum hysteresis compared with
any other cycle having the same range of T. For
simple harmonic cycling of T, the hysteresis predicted
by this model will be frequency dependent.

Suppose T is cycled according to the relation

$$T = T_0 + \Delta T \cos \omega t , \tag{35}$$

then a periodic solution for ϵ satisfying (32) is

$$\epsilon = \frac{T_0}{K_0} + [K_0 + (K_0 + K_1)(\omega_T)^2] \cos \omega t$$

$$+ K_1 \omega_T \sin \omega t [K_0^2 + (K_0 + K_1)^2 (\omega_T)^2]^{-1} , \quad (36)$$

where both T and ϵ are of period $2\pi/\omega$. Since ϵ is out of phase with T, a plot of T vs ϵ as in Fig. 7 shows a characteristic hysteresis loop. Had we allowed for an arbitrary initial condition then the resulting plot of T vs ϵ would also show some creep effect as in Fig. 4, with T vs ϵ ultimately approaching the stable hysteresis loop implied by (35) and (36).

The hysteresis for a closed cycle will be defined by

$$H \equiv \oint T \, d\epsilon . \qquad (37)$$

For the idealized process represented schematically in Fig. 6, the hysteresis is simply

$$H_i = T_A (\epsilon_B - \epsilon_A)$$
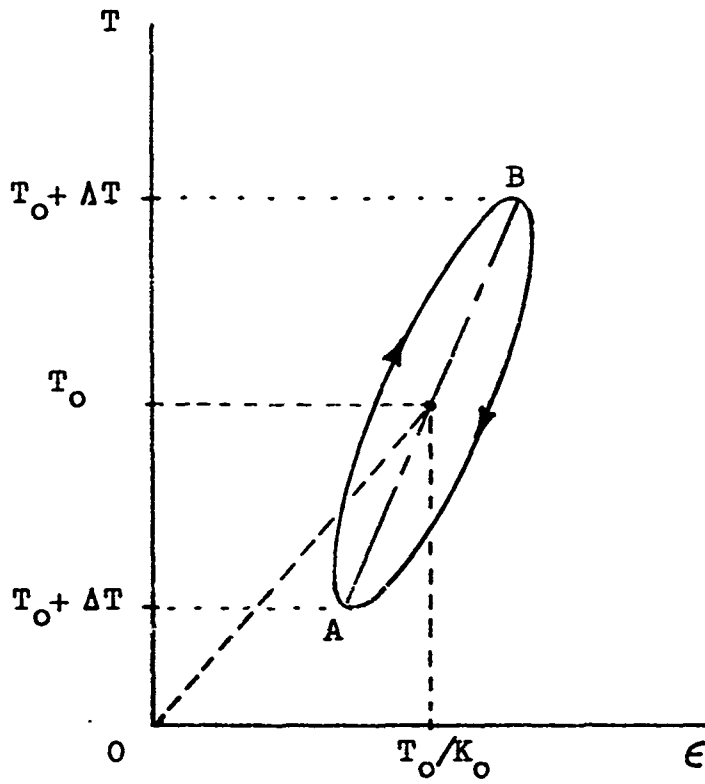
Fig. 7 Hysteresis ellipse for simple harmonic
cycling of the Maxwell material.

or

$$H_i = \frac{4\ K_1\ (\Delta T)^2}{K_0\ (K_0 + K_1)}\ , \qquad (38)$$

where $\Delta T = T_A/2$. For simple harmonic cycling we can compute H by rewriting (37) in the form

$$H = \int_0^{2\pi/\omega} T\ \frac{d\epsilon}{dt}\ dt\ . \qquad (39)$$

Using (35) and (36) this gives

$$H = \frac{\pi\ K_1\ (\Delta T)^2\ \omega\tau}{[K_0^2 + (K_0 + K_1)^2\ (\omega\tau)^2]}\ . \qquad (40)$$

A non-dimensional plot of H versus ω according to this relation is shown in Fig. 8. The latter has a maximum value when $\omega\tau = K_0/(K_0 + K_1)$, the maximum value of H being

$$H_m = \frac{\pi\ K_1\ (\Delta T)^2}{2\ K_0\ (K_0 + K_1)}\ . \qquad (41)$$
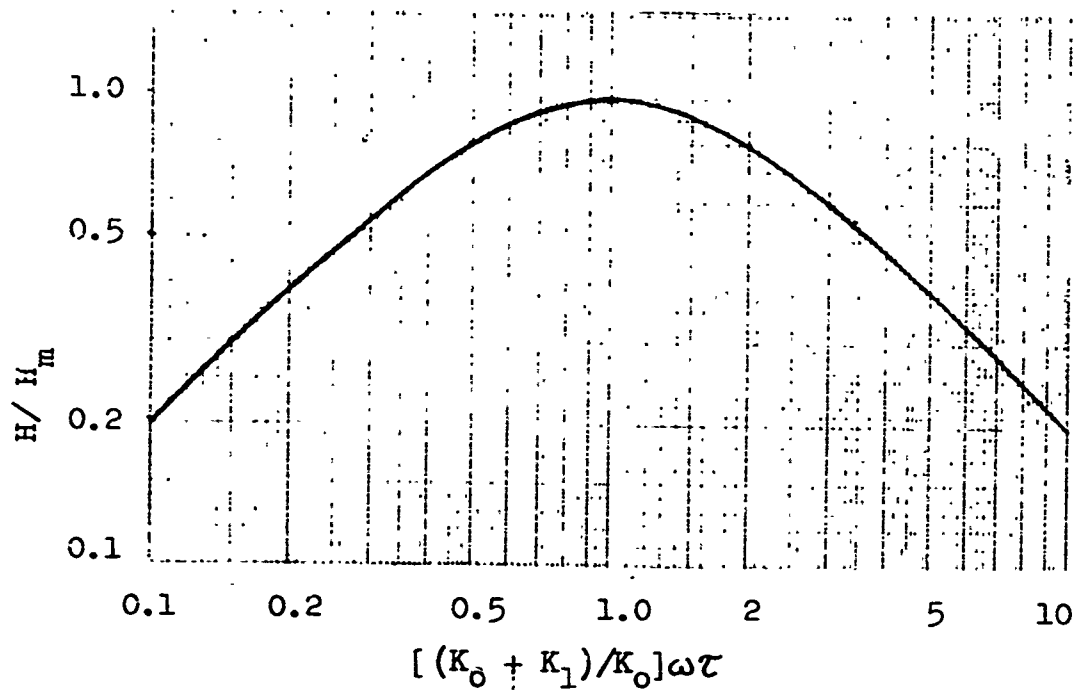
Fig. 8   Relative hysteresis versus a non-dimensional
frequency parameter for simple harmonic
cycling of the Maxwell material at fixed ΔT.

Comparison with (38) indicates that the maximum
hysteresis for simple harmonic cycling is $\pi/8$ times
that for the idea :ed process.

The slope of the major diagonal of the hysteresis
ellipse (line AB, Fig. 7) represents an apparent spring
coefficient K^* whose value can be shown to be given by

$$K^* = \left\{ \frac{K_0{}^2 + (K_0 + K_1)^2 (\omega\tau)^2}{1 + (\omega\tau)^2} \right\}^{1/2} . \tag{42}$$

For $\omega\tau \ll 1$, K^* is nearly equal to K_0 while for
$\omega\tau \gg 1$, K^* approaches $K_0 + K_1$. Thus the apparent
spring coefficient for this model is frequency dependent.

The elastic energy per unit length, E_e, for the
Maxwell model is the sum of that stored in segments AB
and DE (see Fig. 5). This can be expressed in the
form

$$E_e = \frac{1}{2} K_0 \epsilon^2 + \frac{1}{2} K_1 \epsilon_1{}^2 . \tag{43}$$

The elastic strain ϵ_1 can be rewritten as the difference
$\epsilon - \xi$ where ξ, it will be recalled, is the anelastic
strain for segment BC. Thus the elastic energy can be
rewritten as

$$E_e = \frac{1}{2} K_0 \epsilon^2 + \frac{1}{2} K_1 (\epsilon - \xi)^2 \quad . \tag{44}$$

Now consider the partial derivatives of the function $E_e (\epsilon, \xi)$:

$$\partial E_e / \partial \epsilon = (K_0 + K_1) \epsilon - K_1 \xi ,$$
$$\tag{45}$$
$$\partial E_e / \partial \xi = -K_1 \epsilon + K_1 \xi .$$

The above two relations can be shown to be consistent with (32) if and only if

$$\partial E_e / \partial \epsilon = T,$$
$$\tag{46}$$
$$\partial E_e / \partial \xi = -N \, \partial \xi / \partial t,$$

where $N = K_1 \tau$. These relations were employed as a starting point in the analysis by Meixner (1965). They are asserted to be more general than the special Maxwell relation since E_e need not have the form (44) for more general materials. Indeed relations (46) will serve as a useful point of departure in the considerations of the following section.

3. Generalization of the Maxwell Model

Although the above simple model allows for
hysteresis and associated relaxational phenomena, it
is obviously deficient in at least two respects. First,
there is no allowance for permanent elongation. Second,
it assumes constant values of elastic coefficients.
Both of these shortcomings can be overcome by adopting
a scheme similar to that suggested by Wilson (1967).
In essence this scheme distinguishes between the quasi-
elastic strain $\epsilon - \epsilon_o$ and the permanent strain ϵ_o,
in which the latter is a function of the maximum tension
which the material has previously experienced. Morec
for given ϵo, the tension is regarded as a non-linear
function of the quasi-elastic strain. The data further-
more indicate that the form of this dependence is
independent of ϵ_o.

The proposed generalized stress-strain model which
incorporates these features can be obtained in a sys-
tematic manner by starting with a rational generaliza-
tion of the elastic energy function E_e. It will be
understood that E_e is the elastic energy per unit
material coordinate (s). As before we will regard this
as a function of the total strain ϵ and the anelastic

strain. The latter, however, consists of a passive permanent strain ϵ_0 and an active anelastic strain ξ. These are distinguished by the fact, that, under normal conditions, $\partial \xi / \partial t$ can be positive or negative while $\partial \epsilon_0 / \partial t$ cannot be negative unless the material is thermally or mechanically reworked.* Specifically ϵ_0 represents the equilibrium value of ϵ at zero load. The elastic part of the strain is $(\epsilon - \epsilon_0 - \xi)$ and the counterpart of ϵ in the previous model is now $(\epsilon - \epsilon_0)$. With this in mind the proposed generalization for E_e is the following:

$$E_e = f(\epsilon - \epsilon_0) + (K_1 / 2)(\epsilon - \epsilon_0 - \xi)^2 , \qquad (47)$$

in which K_1 is a constant with dimensions of force and $f(\alpha)$ denotes a non-linear, positive function of $\alpha = \epsilon - \epsilon_0$, dependent upon the material and configuration. Relation (44) is a special case in which $\epsilon_0 = 0$ and $f(\epsilon) = (K_0 / 2) \epsilon^2$. Using (46) yields

---

*In essence, this is a manifestation of the second law of thermodynamics as applied to real materials in which ϵ_0 is related closely to the entropy of the material, at least for adiabatic conditions.

$$T = f'(\epsilon - \epsilon_0) + K_1 (\epsilon - \epsilon_0 - \xi) \tag{48}$$

and

$$\tau_1 \frac{\partial \xi}{\partial t} = (\epsilon - \epsilon_0 - \xi) , \tag{49}$$

where $f'(\alpha) \equiv df/d\alpha$ and $\tau_1 = N/K_1$. The latter time constant now carries a subscript since we will introduce at least one more time constant later.

For very slow rate of strain, (49) reduces to the equilibrium relation

$$\epsilon - \epsilon_0 - \xi = 0 \tag{50}$$

and the associated quasi-static load-strain relation becomes

$$T = f'(\epsilon - \epsilon_0). \tag{51}$$

On the other hand, for very rapid rate of straining ξ is small compared with $\epsilon - \epsilon_0$, although $\tau_1 \partial\xi/\partial t$ is not. In this case, (48) reduces to the dynamic load-strain relation

$$T_d = f'(\epsilon - \epsilon_0) + K_1 (\epsilon - \epsilon_0). \tag{52}$$

The associated dynamic spring coefficient is

$$\left(\frac{\partial T}{\partial \epsilon}\right)_d = f''(\epsilon - \epsilon_0) + K_1 . \tag{53}$$

Wilson (1967) has suggested a simple power law dependency of T on $(\epsilon - \epsilon_0)$ for dynamic conditions; this is of the form

$$T_d = Y (\epsilon - \epsilon_0)^n , \tag{54}$$

where Y is a constant. He finds that $n = 3$ for nylon rope and 3/2 for stranded steel cable. However, this relation appears to give a reasonable fit to the data only for moderate to large values of $\epsilon - \epsilon_0$. At small $(\epsilon - \epsilon_0)$ the data pertinent to the dynamic tests tend to indicate a linear dependency of T_d on $(\epsilon - \epsilon_0)$ since $(dT/d\epsilon)_d$ has a finite slope at $\epsilon = \epsilon_0$ (see Fig. 3).

An alternative is to take $f(\alpha)$ proportional to αn such that T_d reduces to a linear relation at small $\epsilon - \epsilon_0$. However, as will be shown, the data of Paquette

and Henderson (1965) indicates that an exponential form
for $f(\alpha)$ is more appropriate, at least for nylon rope.
They evaluated the dynamic spring coefficient at ten
mean tensions from the slope of the stabilized hysteresis
loops (of small ΔT) for the test illustrated in Fig. 4.
Such measurements are potentially more accurate than
inferences drawn from the large hysteresis loops
analysed in Wilson's study. Their results are shown
as a function of tension in Fig. 9. These seem to indicate
a linear dependency of $(\partial T/\partial \epsilon)_d$ on T. A least squares
fit of the form

$$(\frac{\partial T}{\partial \epsilon})_d = K + bT,$$

yields   (55)

$$K = 1.4 \times 10^4 \text{ lbs}$$

$$b = 26.0 ,$$

with a standard error of estimate of $(\partial T/\partial \epsilon)_d$ of
about 0.7×10^4 lbs for the 1/2 inch diameter nylon rope.
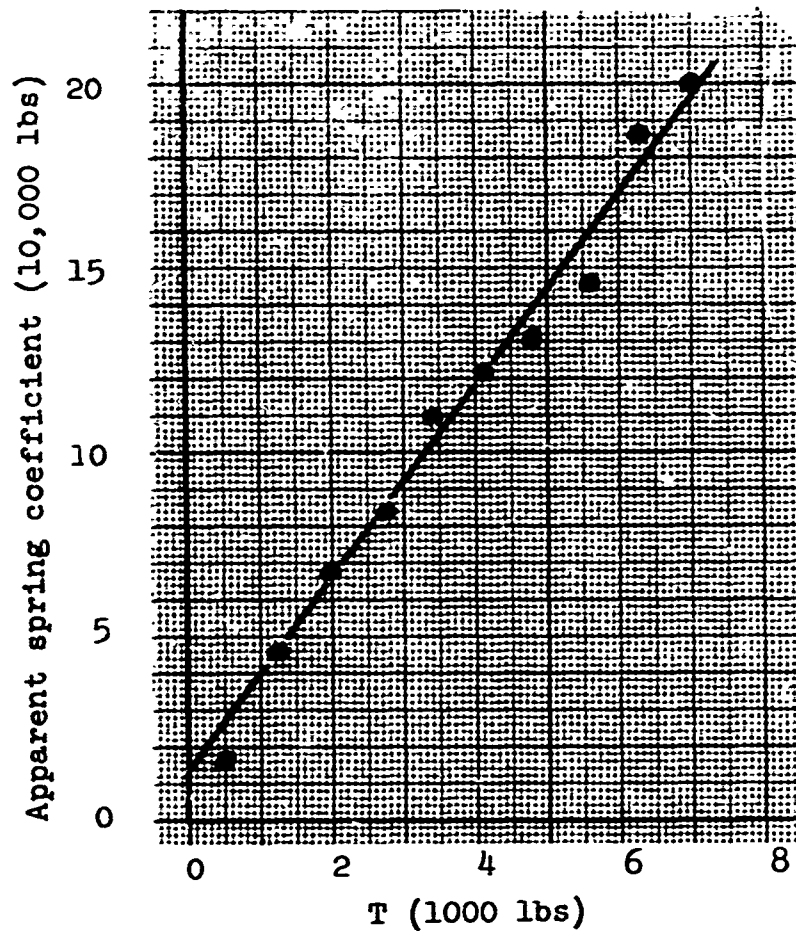
Fig. 9   Apparent spring coefficient for one-half
         inch diameter nylon rope versus mean tension,
         based upon tests by Paquette and Henderson
         (1965) at ± 180 lb cycling amplitude; full
         line represents a least square linear
         regression.

If we accept the empirical fit (55) then from (52), (53), and (55) we get

$$f''(\alpha) - b\, f'(\alpha) = (K - K_1) + b\, K\, \alpha, \tag{56}$$

where $\alpha \equiv \epsilon - \epsilon_0$. This leads to a solution for $f'(\alpha)$ of the form

$$f'(\alpha) = c\, e^{b\alpha} - \frac{K}{b} - K_1\, \alpha, \tag{57}$$

where c is a constant of integration. Thus from (52) we get

$$T_d = c\, e^{b\,(\epsilon - \epsilon_0)} - K/b, \tag{58}$$

however, T_d must vanish when $\epsilon = \epsilon_0$ (by definition of ϵ_0) and hence we require that $c = K/b$. Consequently

$$T_d = \frac{K}{b} \left[e^{b\,(\epsilon - \epsilon_0)} - 1 \right]. \tag{59}$$

The limiting slope at zero $\epsilon - \epsilon_0$ (zero T_d) is

$$(\partial T_d / \partial \epsilon)_0 = K. \tag{60}$$

Moreover for the static case, for which T is given by (51), we get

$$T = T_d - K_1 (\epsilon - \epsilon_0) . \tag{61}$$

This has a limiting slope at zero $\epsilon - \epsilon_0$ given by

$$(\partial T/\partial \epsilon)_0 = (K - K_1) , \tag{62}$$

which clearly requires that $K_1 \le K$.

Finally from (47) and (57) with $c = K/b$ we get for the elastic energy

$$E_e = \frac{K}{b^2} [e^{b(\epsilon - \epsilon_0)} - b (\epsilon - \epsilon_0)]$$
$$- K_1 (\epsilon - \epsilon_0)\xi + (K_1/2) \xi^2 . \tag{63}$$

For very small $(\epsilon - \epsilon_0)$ this relation reduces to that for the simple Maxwell system where K is equivalent to $K_0 + K_1$ for that system.

As an application of (59), with the coefficients given by (55), we note that a dynamic tension of 7000 lbs will give rise to a quasi-elastic elongational strain $(\epsilon - \epsilon_0)$ of about ten per cent. However, under quasi-static conditions the elongation would be significantly

larger, particularly if K_1 is nearly equal to K.

For the general straining conditions, (48) and (57) yield

$$T = \frac{K}{b} [e^{b(\epsilon - \epsilon_0)} -1] - K_1 \xi , \qquad (64)$$

which is consistent with $\partial E_e/\partial \epsilon$ from (63). This together with the prognostic relation for ξ, Eq (49), represents a generalized model for the case of constant ϵ_0.

4. The "Saturation" Relation

To relations (64) and (49) we must add a third relation governing ϵ_0. Figs. 3 and 4 indicate that for a given ϵ_0 there apparently exists a sort of "saturation" curve in the T, ϵ-diagram. This corresponds to the path which would result for very slow undirectional straining from the original state to the ultimate point (path O S U of Fig. 10). It is strictly an irreversible one-way path, for if the load is removed at some inter- mediate point S, even very slowly, the strain will return to some value ϵ_0 along a different path SB.
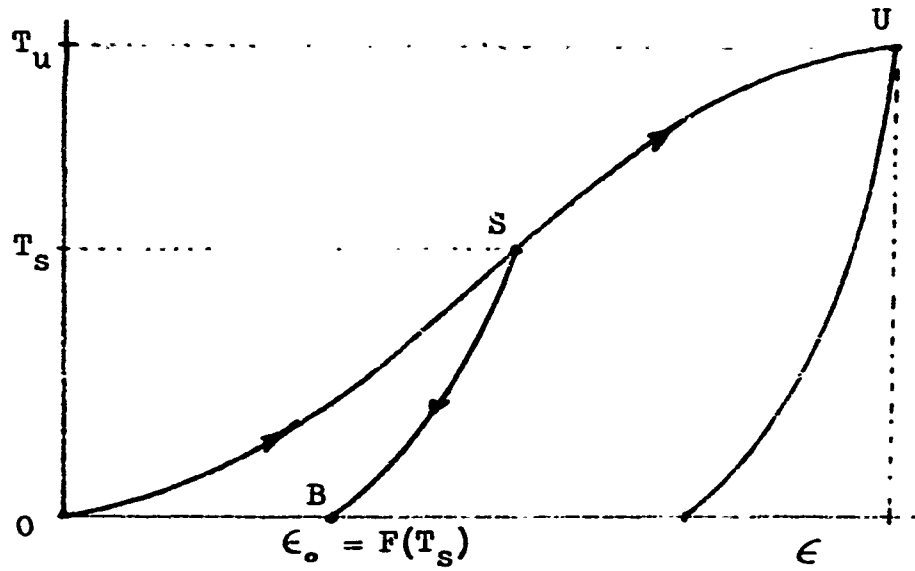
Fig. 10   Schematic illustrating the "saturation" curve OSU in the load-strain diagram.
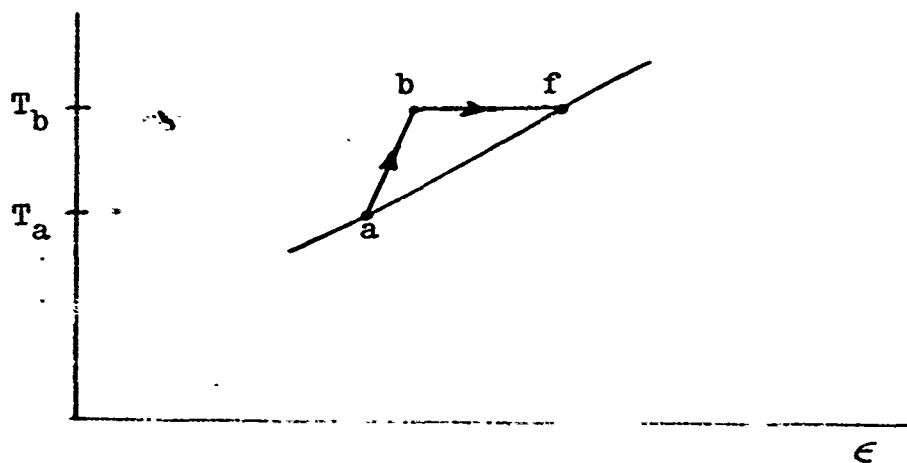
Fig. 11   Schematic of dynamic overloading in the vicinity of the "saturation" curve.

Thus in principle

$$\epsilon_o = F(T_s) , \qquad (65)$$

the form of the function being readily deduced empirically. The data summarized by Wilson suggest that a power law form

$$F(T_s) \alpha T_s^{m} \qquad (66)$$

leads to a reasonable fit, at least for tensions less than 75 per cent of the ultimate value. For ropes $m < 1$ while for steel cable $m > 1$.

It is not implied by the above discussion that T_s is an upper bound for T for given ϵ_o. The actual T versus ϵ near the saturation level depends upon the rate of strain. For example, referring again to Fig. 4, the relation T_s versus ϵ is shown by the dashed line drawn through the upper ends of the final hysteresis loops. There is an "overshoot" of this saturation curve which occurs when the mean load is suddenly increased after having reached a stable hysteresis loop. In order to simulate this effect, the following relation is suggested for ϵ_o :

$$\tau_0 \; \partial\epsilon_0/\partial t \;\; = \begin{cases} F\,(T) \,-\, \epsilon_0, & \text{if} \quad \epsilon_0 \,<\, F(T) \\[2mm] 0 \quad, & \text{if} \quad \epsilon_0 \,\geq\, F(T) \end{cases} \qquad (67)$$

where τ_0 is a second characteristic relaxation time for the rope or cable. If T is initially at the saturation level for given ϵ_0 and if $\partial T/\partial t > 0$, but of very small magnitude, then ϵ_0 will change nearly in accord with $F\,(T)$.

Suppose, on the other hand, that T is initially at some saturation level T_a (Fig. 11), corresponding to ϵ_{0a}, but experiences a sudden increase to the value T_b after which it remains constant. Then since $\epsilon_{0a} < F(T_b)$, ϵ_0 must increase with time approaching the final value $\epsilon_{0f} = F(T_b)$ asymptotically. Specifically relation (67) implies that

$$\epsilon_0 \;=\; \epsilon_{0f} \,-\, (\epsilon_{0f} \,-\, \epsilon_{0a})\; e^{-t/\tau_0} \qquad (68)$$

for this case. The change of ϵ will have essentially the same behavior. As a test of this relation, the data of Fig. 4 was analysed for the case where the mean tension was suddenly increased from about 6300 lbs to 7000 lbs. In this test T was also varied periodically over the range 7000 ± 180 lbs after the increase, thus giving a progressive series of loops which provide a time
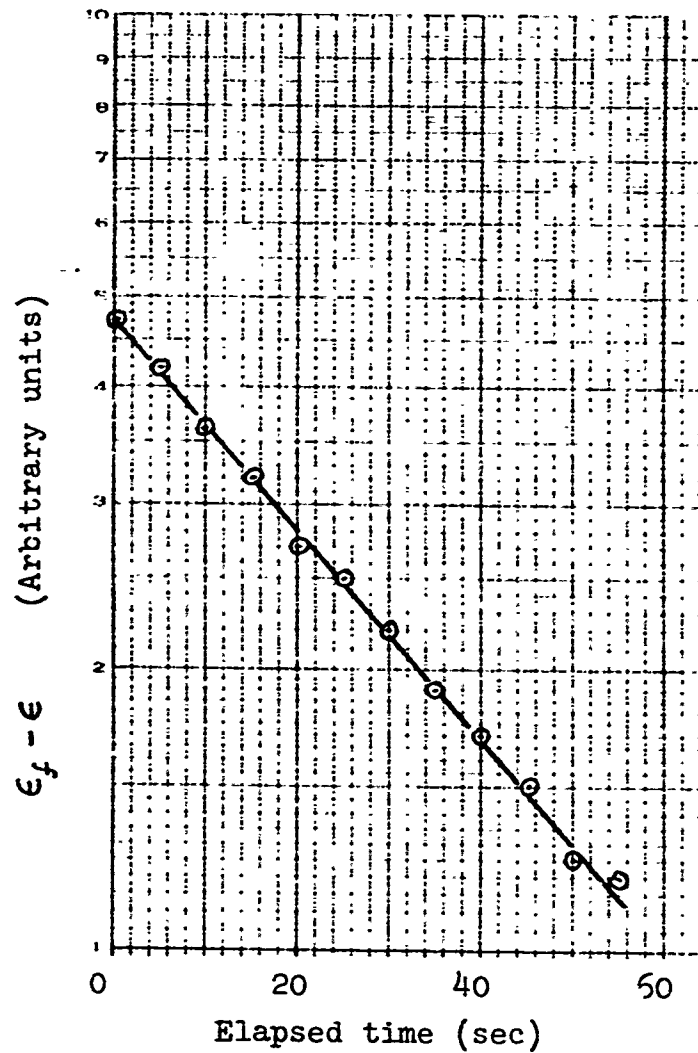
Fig. 12 Relative strain (log scale) versus elapsed
time, illustrating the creep phenomenon
near the "saturation" curve (based upon
data of Paquette and Henderson, 1965).

scale for the creep phenomenon. The period of cycling for this case, as deduced from the data of Paquette and Henderson, is about ten seconds. A semi-log plot of $\epsilon_{of} - \epsilon_o$ (in relative units) versus time is shown in Fig. 12. This clearly confirms the exponential relaxation and yields a value of τ_o for the nylon line of about 40 seconds.

5. Hysteresis for Nylon Rope

The measurements of Paquette and Henderson (1965) for the one-half inch diameter nylon rope included evaluation of the hysteresis for five different ranges of ΔT all at a common mean tension of 2000 lbs and at cycling periods from 12 to about 60 seconds. Their results indicated that the hysteresis energy absorption per cycle, H, was nearly independent of the period of cycling. The results are summarized in Table 1.

In terms of the present load-strain model, the lack of dependency of H on the period of cycling should imply that H is near its maximum, so that it is linearly independent of ω.

TABLE 1

HYSTERESIS IN HALF-INCH NYLON ROPE
AT MEAN TENSION OF 2000 lbs

(After ~aquette and Henderson, 1965)

Tension Variation ΔT (lb)	Hysteresis per cycle (lb)	Period of Cycling (sec)
± 140	0.12	18, 36
± 280	0.52	12, 16, 40, 57
± 560	2.3	18, 30
± 1120	15.2	30, 50
± 1400	29.7	30, 55

For small amplitude cycling about a mean tension
T_0, the linear theory of section 2 should be applicable
provided that $(K_0 + K_1)$ is taken as the dynamic spring
coefficient for the mean tension T_c. From Fig. 9 with
$T_0 = 2000$lbs we get $(K_0 + K_1) = 6.8 \times 10^4$ lbs. From
the results of section 3 it was found that K_1 should
not exceed 1.4×10^4 lbs for this same case. If we
consider that H is at its maximum given by (41) and
that $K_1 = 1.4 \times 10^4$ lbs then we find that at most

$$H = 6.0 \times 10^{-6} \ (\Delta T)^2 \ , \tag{69}$$

for small ΔT, in the case of the one-half inch nylon
line tested by Paquette and Henderson. A plot of the
data from Table 1 is shown in Fig. 13. The full curve
is simply a smooth interpolation. The dashed curve
is a plot of relation (69). The fact that it is
asymptotic to the full curve for small ΔT seems to
be a verification of the assumptions leading to relation
(69); namely that H is at its maximum for given ΔT
and that K_1 is the maximum permissible value consistent
with the data of Fig. 9. If this is correct then $\overline{\omega}\tau_1$
should be equal to the value for which H has it maximum,
namely $K_0/(K_0 + K_1)$ which has the magnitude of 0.79.
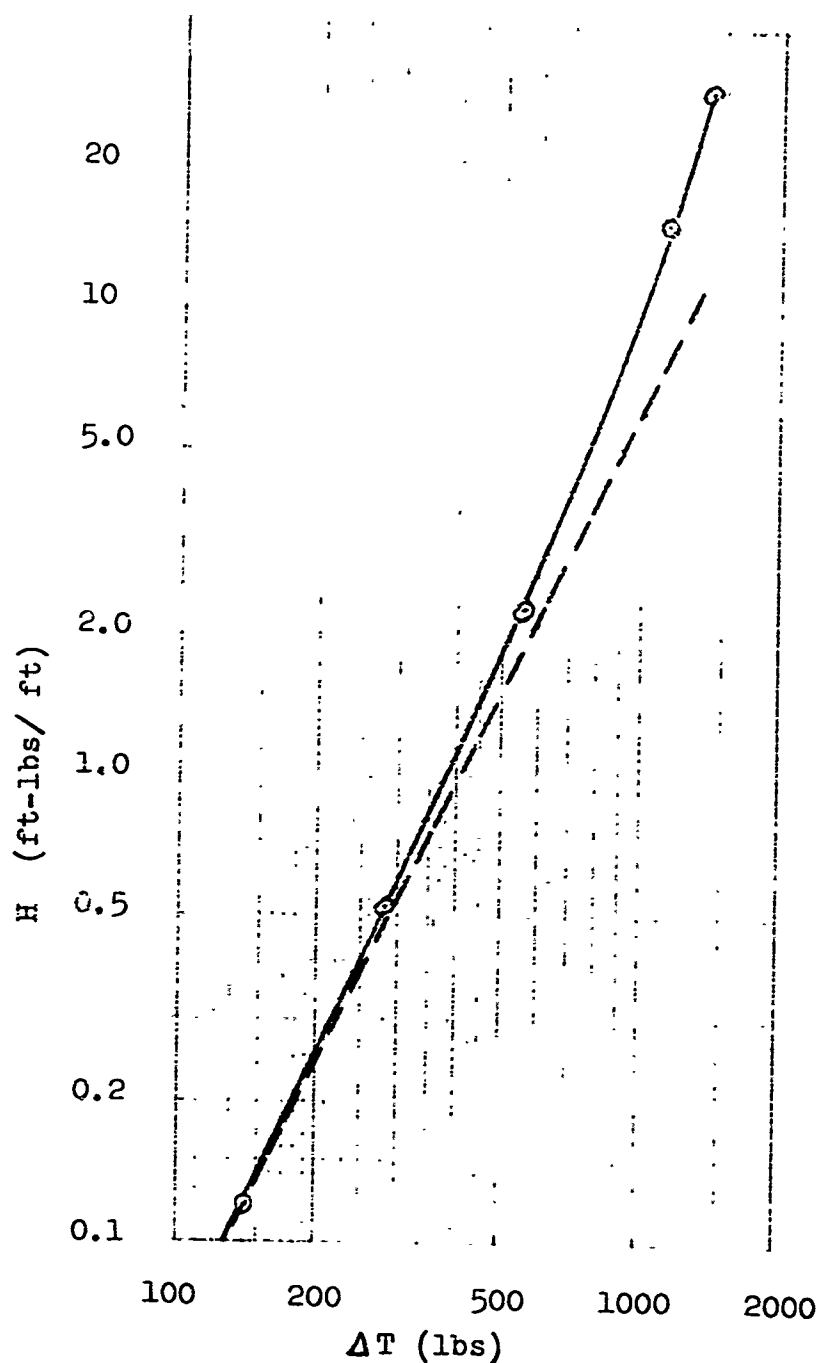
Fig. 13 Hysteresis (H) versus amplitude of cyclic
loading (ΔT) for one-half inch diameter
nylon rope; circled points represent
measurements by Paquette and Henderson
(1965); dashed line is based upon the
model discussed in the text.

The overall mean period of cycling for these tests was about 30 seconds and hence τ_1 is presumably about four seconds.

There are, however, two disturbing features of the data which are not adequately explained by the present model. First of all, the appreciable departure of H at large ΔT from that given by (69) is not adequately explained. Second, that data of Paquette and Henderson indicate that the dynamic spring coefficient decreases with increasing ΔT for fixed value of mean tension. The non-linear aspect of the load-strain relation does not adequately explain these observations. A possible explanation may lie in the frequency dependence of H and the apparent spring coefficient. However, adequate tests of the frequency dependence are lacking.

6. Summary and Possible Further Generalization

At least many of the observed properties of ropes and cables seem to be in accord with the foregoing model which is summarized here:

$$T = \frac{K}{b} \left[e^{b(\epsilon - \epsilon_0)} + 1 \right] - K_1 \xi , \qquad (69)$$

$$\tau_1 \frac{\partial \xi}{\partial t} = (\epsilon - \epsilon_0 - \xi) , \qquad (70)$$

$$\tau_o \frac{\partial \epsilon_o}{\partial t} = \begin{cases} F(T) - \epsilon_o \,, & \text{if } \epsilon_o < F(T) \\ 0 & , \text{if } \epsilon_o \geq F(T) \end{cases} \qquad (71)$$

where $F(T) = \epsilon_o$ corresponds to the saturation relation appropriate to very slow unidirectional straining.

It must be emphasized that further critical measurements are needed to check the validity of the model, particularly in respect to the effect of time rate of strain. Measurements of hysteresis over a wide range of cycling periods are required for an accurate evaluation of the parameters τ_1 and K_1 for a particular rope or cable. Indeed it is quite conceivable that the true behavior of ropes and cables requires a more complex model, incorporating a discrete spectrum of relaxation times and associated K values.

In order to clarify the above remark we note that relations (69) and (70) can be recast in the following integral form in which ξ has been eliminated

$$T = \frac{K}{b} (e^{b\alpha} - 1) - \frac{K_1}{\tau_1} \int_0^\infty \alpha(t - \lambda) e^{-\lambda/\tau_1} \, d\lambda \,, \qquad (72)$$

where $\alpha \equiv \epsilon - \epsilon_o$. The integral is a particular convolution of the function $\alpha(t)$. A natural generalization of this relation is

$$T = \frac{K}{b} (e^{b\alpha} - 1) - \int_0^\infty \alpha(t - \lambda) G(\lambda) d\lambda , \qquad (73)$$

in which the generalized function $G(\lambda)$ can be represented in the form

$$G(\lambda) = \sum_{j=1}^{n} (K_j/\tau_j) e^{-\lambda/\tau_j} . \qquad (74)$$

Relation (72) is a special case in which $n = 1$. Clearly (73) has greater flexibility in respect to fitting the observed frequency dependence of hysteresis in the case of cyclic loading. For small amplitude, simple harmonic variation of T or ϵ at fixed ϵ_0 and frequency ω, relation (73) can be shown to yield the following hysteresis

$$H = \frac{\pi h (\omega) (\Delta T)^2}{K^*} \qquad (75)$$

in which $h (\omega)$ is the transfer function

$$h (\omega) = \int_0^\infty G(\lambda) \sin \omega\lambda \, d\lambda \qquad (76)$$

and K^* is the dynamic spring coefficient. Using (74) gives

$$h(\omega) = \sum_{j=1}^{n} K_j \; \frac{\omega\tau_j}{[1 + (\omega\tau_j)^2]} \; . \tag{77}$$

As an example, suppose $n = 2$ with

$$K_1 = K_2 \quad \text{and} \quad \tau_2 = 10\,\tau_1 .$$

In this case the function $h(\omega)$ possesses a very broad range of ω over which it is nearly a constant (Fig. 14). There is some indirect evidence that this behavior may be closer to reality than that implied by the simpler $(n = 1)$ model. However, lacking really definitive evidence for this, we will adopt the model delineated by (69) to (70) in the remainder of this study.
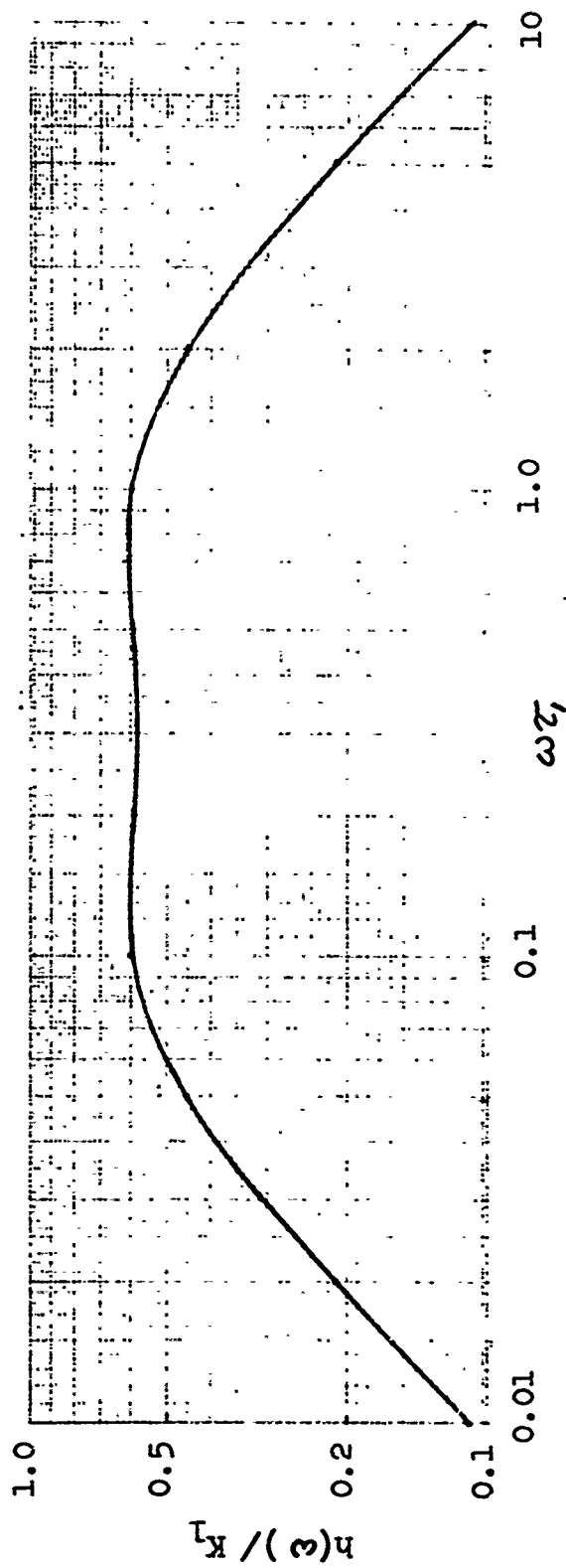
Fig. 14 Relative transfer function versus frequency for the generalized visco-elastic model governed by (73) and (74) for the case $n = 2$, $K_1 = K_2$, and $\tau_2 = 10\,\tau_1$.

IV.   ENERGY CONSIDERATIONS

1.   Energy Equation for a Mooring Line

The equation of motion of a mooring line in vector form was given by (15). If we form the scalar produc of $\hat{V}$ with this equation we obtain

$$\frac{\partial}{\partial t} \left[\frac{1}{2} (m + m_a) \, |\hat{V}|^2 - \frac{1}{2} m_a V_T^2 \right]$$

$$= \hat{V} \cdot \frac{\partial \hat{T}}{\partial s} - w \, \hat{k} \cdot \hat{V} + \hat{R} \cdot \hat{V} , \tag{78}$$

where for simplicity we will regard m_a and w as constants. The term $w \, \hat{k} \cdot \hat{V}$ can be written as $\partial(wz)/\partial t$ where z is the elevation of a material point of the line at s, t and wz is an effective potential energy in the water per unit of s.

Now consider the term $\hat{V} \cdot \partial \hat{T}/\partial s$. This can be written as

$$\hat{V} \cdot \frac{\partial \hat{T}}{\partial s} = \frac{\partial}{\partial s} (\hat{V} \cdot \hat{T}) - \hat{T} \frac{\partial \hat{V}}{\partial s} . \tag{79}$$

Furthermore, since $\hat{V} = \partial\hat{r}/\partial t$ we can write

$$\frac{\partial\hat{V}}{\partial s} = \frac{\partial}{\partial t}\left(\frac{\partial\hat{r}}{\partial s}\right)$$

or

$$\frac{\partial\hat{V}}{\partial s} = \frac{\partial}{\partial t}\left(\hat{\tau}\,\frac{\partial\sigma}{\partial s}\right)$$

using (22). Now $\partial\sigma/\partial s = 1 + \epsilon$ where ϵ is the elongational strain at s, t. Hence we find

$$\frac{\partial\hat{V}}{\partial s} = \hat{\tau}\,\frac{\partial\epsilon}{\partial t} + (1 + \epsilon)\,\frac{\partial\hat{\tau}}{\partial t} \ . \tag{80}$$

Moreover $\hat{T} = T\,\hat{\tau}$ and since $\partial\hat{\tau}/\partial t$ is orthogonal to $\hat{\tau}$, it follows that

$$\hat{T}\cdot\frac{\partial\hat{V}}{\partial s} = T\,\frac{\partial\epsilon}{\partial t} \ . \tag{81}$$

Thus from (79) and (81), the energy equation (78) takes the form

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}(m + m_a)\,V^2 - \frac{1}{2}\,m_a\,V_\tau^2 + wz\right\}$$

$$+ T\,\frac{\partial\epsilon}{\partial t} = \frac{\partial}{\partial s}(\hat{V}\cdot\hat{T}) + \hat{R}\cdot\hat{V} \ , \tag{82}$$

where V is simply the magnitude of $\hat{V}$. It will be recalled from the previous chapter that the elastic energy E_e is a function of at least three different strains (ϵ, ξ, and ϵ_0). It follows therefore that

$$\frac{\partial E_e}{\partial t} = \frac{\partial E_e}{\partial \epsilon} \frac{\partial \epsilon}{\partial t} + \frac{\partial E_e}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial E_e}{\partial \epsilon_0} \frac{\partial \epsilon_0}{\partial t} \ . \tag{83}$$

However, since ϵ and ϵ_0 enter only in the form $(\epsilon - \epsilon_0)$ then $\partial E_e / \partial \epsilon_0 = - \partial E_e / \partial \epsilon$. Also using (46) we find

$$\frac{\partial E_e}{\partial t} = T \frac{\partial \epsilon}{\partial t} - N \left(\frac{\partial \xi}{\partial t}\right)^2 - T \frac{\partial \epsilon_0}{\partial t} \ . \tag{84}$$

Consequently the energy equation finally takes the form

$$\frac{\partial}{\partial t} (E_k + E_p + E_e) = \frac{\partial}{\partial s} (\hat{V} \cdot \hat{T}) + \hat{R} \cdot \hat{V}$$

$$- N \left(\frac{\partial \xi}{\partial t}\right)^2 - T \frac{\partial \epsilon_0}{\partial t} \ , \tag{85}$$

in which

$$E_k = \frac{1}{2} m V^2 + \frac{1}{2} \dot{m}_a V_n^2$$

(86)

$$E_p = w z ,$$

where $V_n^2 = V^2 - V_\tau^2$. Each of these energy functions represents energy per unit material coordinate (s). The kinetic energy includes that of the mooring line plus that of the effective added mass of water, which is related to the normal component of motion of the line.

2. Energy Supply and Loss for the Line

If we integrate (85) with respect to s from the anchor point to the point of attachment on the surface vessel we obtain

$$\frac{\partial}{\partial t} \int_0^L E\, ds = (\hat{V} \cdot \hat{T})_L + \int_0^L \hat{R} \cdot \hat{V}\, ds$$
$$- \int_0^L N \left(\frac{\partial \varepsilon}{\partial t}\right)^2 ds - \int_0^L T \frac{\partial \varepsilon_0}{\partial t}\, ds, \qquad (87)$$

where $\int_0^L E\, ds$ is the total mechanical energy of the line (kinetic, potential, and elastic energy). The term $(\hat{V} \cdot \hat{T})_L$ represents the rate of work done on the mooring line by the ship. The integral involving $\hat{R} \cdot \hat{V}$ is the

net work done by the water on the mooring line per unit
time. The remaining two terms represent rates of energy
loss (or more correctly, rates of energy conversion
from mechanical to unavailable thermal form). They are
related to the non-equilibrium processes occurring within
the line (hysteresis and permanent strain). The latter
terms are always of one sign for clearly neither $(\partial\xi/\partial t)^2$
nor $\partial\epsilon_0/\partial t$ can be negative. However, $\partial\epsilon_0/\partial t$ can be
zero if the range of T for given ϵ_0 is suitably
limited. On the other hand $(\partial\xi/\partial t)^2$ can vanish only
under static conditions.

If the water is nominally at rest $(\hat{U} = 0)$, then
the term $\hat{R}\cdot\hat{V}$ is negative and represents a rate of
energy loss to the water by form drag. In this case
a steady state of dynamic energy of the line can be
maintained only if $(\hat{V}\cdot\hat{T})_L$ is positive. Clearly in the
deep water where $\hat{U}$ is negligible $\hat{R}\cdot\hat{V}$ will indeed
tend to be negative and hence lead to damping of trans-
verse oscillations in that region of the line. The
energy loss terms related to $(\partial\xi/\partial t)^2$ and $\partial\epsilon_0/\partial t$,
on the other hand, are effective in damping longitudinal
waves in the line.

It can be shown that for cyclic variation of T the mean value of $N(\partial\xi/\partial t)^2$ is simply H/P, where P is the period of cycling and H is the hysteresis defined by (37).

V. MOORING LINE EQUATIONS IN CHARACTERISTIC FORM

1. Geometrical Considerations

Let x, y, z be a fixed Cartesian coordinate system
with origin at the anchor point and with z representing
elevation. Let $\hat{i}, \hat{j}, \hat{k}$ represent the associated unit
vectors for this fixed coordinate system. The natural
coordinates are defined in terms of the local orientation
of the mooring line at material point s at time t
(Zajac, 1957). It will be recalled that $\hat{\tau}$ is a unit
vector tangent to the axis of the mooring line at s, t
and in the direction of increasing s. Now let $\hat{\mu}$ be a
unit vector normal to $\hat{k}$ and $\hat{\tau}$ with the direction of
$\hat{k} \times \hat{\tau}$. Moreover, let $\hat{\nu}$ be a unit vector normal to $\hat{\tau}$
and $\hat{\mu}$ with direction $\hat{\mu} \times \hat{\tau}$. Thus $\hat{\nu}, \hat{\mu}, \hat{\tau}$ form a
mutually orthogonal, right handed, set of local coordinate
vectors.

Let θ denote the zenith angle and ϕ the azimuth*
angle of the local unit vector $\hat{\tau}$ relative to the fixed
Cartesian coordinates (see Fig. 15). Note that these
angles are <u>not</u> coordinate angles of the position vector

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* More precisely ϕ is the azimuth of the vertical
plane containing vector τ, as measured from the x axis.
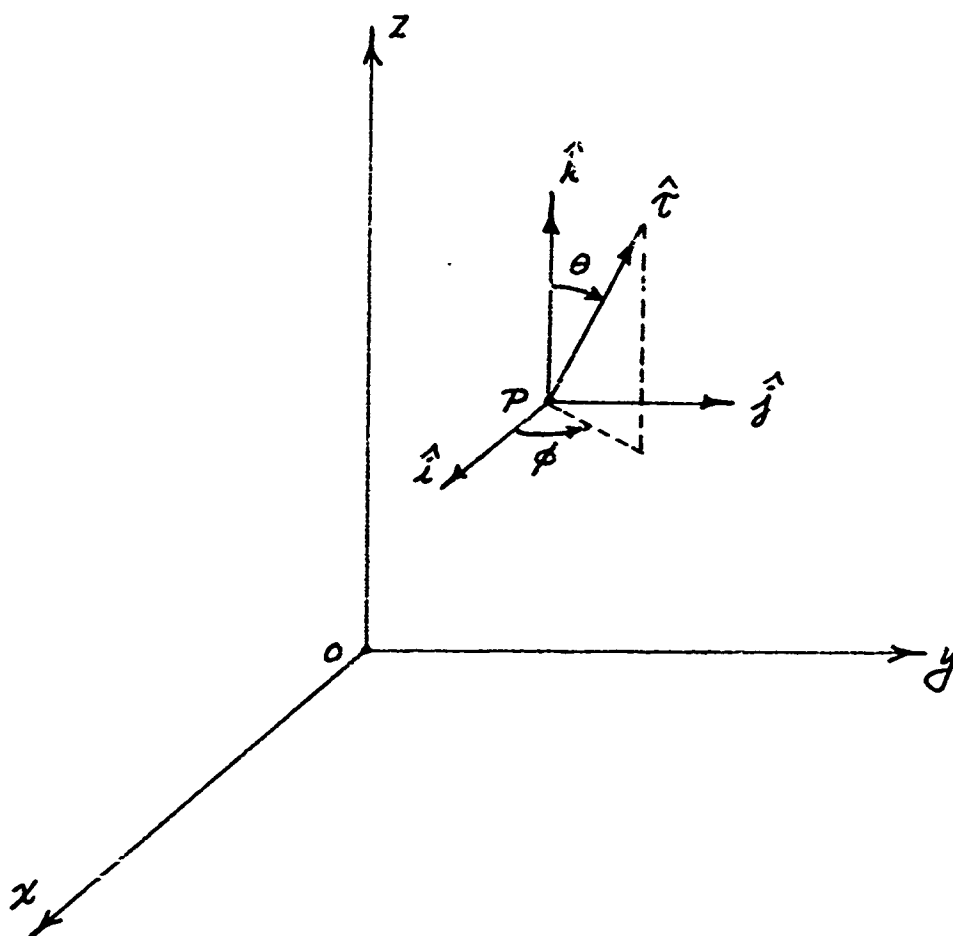
Fig. 15   Schematic illustrating the local coordinate system for a mooring line and its relation to a fixed Cartesian frame of reference.

of P but rather of the orientation of the tangent at point P. From the geometry it can be shown that the relations between the local coordinate vectors and the fixed reference coordinate vectors are given by

$$\hat{\nu} = \hat{i} \cos \theta \cos \Phi + \hat{j} \cos \theta \sin \Phi - \hat{k} \sin \theta,$$

$$\hat{\mu} = -\hat{i} \sin \Phi + \hat{j} \cos \Phi , \qquad (88)$$

$$\hat{\tau} = \hat{i} \sin \theta \cos \Phi + \hat{j} \sin \theta \sin \Phi + \hat{k} \cos \theta .$$

The orientation of $\hat{\nu}, \hat{\mu}, \hat{\tau}$ in general vary with s along the line at given t or with t at fixed s. Using relations (88), it is easily established that

$$d\hat{\nu} = \hat{\mu} \cos \theta \, d\Phi - \hat{\tau} \, d\theta ,$$

$$d\hat{\mu} = -\hat{\nu} \cos \theta \, d\Phi - \hat{\tau} \sin \theta \, d\Phi , \qquad (89)$$

$$d\hat{\tau} = \hat{\nu} \, d\theta + \hat{\mu} \sin \theta \, d\Phi .$$

2. Kinematical Relations Involving the Dependent Variables

The vector $\hat{V}$, which represents the velocity of a material point of the mooring line, can be represented in the form

$$\hat{V} = V_\nu \, \hat{\nu} + V_\mu \, \hat{\mu} + V_\tau \, \hat{\tau} , \qquad (90)$$

in which V_ν and V_μ are normal components and V_τ is the tangential component of the velocity. The acceleration of a material point is given by

$$\frac{\partial \hat{v}}{\partial t} = \frac{\partial V_\nu}{\partial t} \hat{\nu} + \frac{\partial V_\mu}{\partial t} \hat{\mu} + \frac{\partial V_\tau}{\partial t} \hat{\tau}$$
$$+ V_\nu \frac{\partial \hat{\nu}}{\partial t} + V\mu \frac{\partial \hat{\mu}}{\partial t} + V_\tau \frac{\partial \hat{\tau}}{\partial t} . \tag{91}$$

However, the turning vectors like $\partial \hat{\mu}/\partial t$ can be evaluated in terms of $\partial \theta/\partial t$ and $\partial \phi/\partial t$ using (89). The resulting relations for the acceleration is

$$\frac{\partial \hat{v}}{\partial t} = \frac{\partial V_\nu}{\partial t} \hat{\nu} + \frac{\partial V_\mu}{\partial t} \hat{\mu} + \frac{\partial V_\tau}{\partial t} \hat{\tau}$$
$$+ V_\nu \left(\cos \theta \frac{\partial \phi}{\partial t} \hat{\mu} - \frac{\partial \theta}{\partial t} \hat{\tau}\right)$$
$$+ V_\mu \left(- \cos \theta \frac{\partial \phi}{\partial t} \hat{\nu} - \sin \theta \frac{\partial \phi}{2t} \hat{\tau}\right)$$
$$+ V_\tau \left(\frac{\partial \theta}{\partial t} \hat{\nu} + \sin \theta \frac{\partial \phi}{\partial t} \hat{\mu}\right) . \tag{92}$$

This relation will be useful for recasting the vector equation of motion into the natural component form.

A relation entirely similar to (92) can be obtained
for $\partial \hat{V}/\partial s$, in which the time derivative terms are re-
placed by derivatives with respect to s. However, we
have already found that $\partial \hat{V}/\partial s$ is related to $\partial \epsilon/\partial t$ and
$\partial \hat{\tau}/\partial t$ by relation (80). If we now write $\partial \hat{\tau}/\partial t$ using
the last of relations (89) and equate coefficients of
corresponding unit vectors in the two different expressions
for $\partial \hat{V}/\partial s$, we find the following (prognostic) relations
for θ, Φ, and ϵ:

$$(1 + \epsilon) \frac{\partial \theta}{\partial t} = \frac{\partial V_\nu}{\partial s} + V_\tau \frac{\partial \theta}{\partial s} - V_\mu \cos \theta \frac{\partial \Phi}{\partial s} , \quad (93)$$

$$(1 + \epsilon) \sin \theta \frac{\partial \Phi}{\partial t} = \frac{\partial V_\mu}{\partial s} + (V_\nu \cos \theta$$

$$+ V_\tau \sin \theta) \frac{\partial \Phi}{\partial s} , \quad (94)$$

$$\frac{\partial \epsilon}{\partial t} = \frac{\partial V_\tau}{\partial s} - V_\mu \sin \theta \frac{\partial \Phi}{\partial s} - V_\nu \frac{\partial \theta}{\partial s} . \quad (95)$$

These are purely kinematical compatability relations
among the dependent variables θ, Φ, ϵ, V_ν, V_μ, and V_τ.
Moreover they are relevant only for the transient state;
in the case of equilibrium conditions they are satisfied
trivially.

Relation (95) can be interpreted in the following way. Suppose we solve for $\partial V_\tau / \partial s$ and integrate over the full range of s. This yields (since $V_\tau (o,t) = 0$):

$$V_\tau (L, t) = \int_0^L \frac{\partial \varepsilon}{\partial t} \, ds + \int_0^L (V_\mu \sin \theta \frac{\partial \phi}{\partial s} + V_\nu \frac{\partial \theta}{\partial s}) \, ds. \tag{96}$$

As a special case, suppose $\partial \theta / \partial s$ and $\partial \phi / \partial s$ vanish. This implies that the line is straight and hence (96) implies that a positive tangential velocity can exist at the end of the line only if the line stretches. On the other hand if the line is curved, then part of V_T at the end of the line can be due to the effect of "taking up slack" in the line; this is represented by the second integral in (96).

Relations (93) and (94) can be interpreted in a somewhat analogous manner. Specifically the line can have a normal component of motion at its upper extremity if the line has a progressive tilt with time or if it experiences a flexing.

The remaining kinematical relations of importance
are those relating the Cartesian coordinates of a
material-point of the line to the natural coordinates.
First of all we note that

$$\frac{\partial \hat{r}}{\partial t} = \hat{V} , \tag{97}$$

where $\hat{V}$ is expressed in natural coordinates by (90) and

$$\hat{r} = x \hat{i} + y \hat{j} + z \hat{k}. \tag{98}$$

Using (88), (90), (97), and (98) we find

$$\frac{\partial x}{\partial t} = V_\nu \cos \theta \cos \Phi - V_\mu \sin \Phi + V_\tau \sin \theta \cos \Phi, \tag{99a}$$

$$\frac{\partial y}{\partial t} = V_\nu \cos \theta \sin \Phi + V_\mu \cos \Phi + V_\tau \sin \theta \sin \Phi, \tag{99b}$$

$$\frac{\partial z}{\partial t} = -V_\nu \sin \theta + V_\tau \cos \theta . \tag{99c}$$

We can also employ (22) to obtain relations for the
derivatives of x, y, and z with respect to s. Using
(22) and (88) yields

$$\frac{\partial x}{\partial s} = (1 + \epsilon) \sin \theta \cos \Phi \; , \tag{100a}$$

$$\frac{\partial y}{\partial s} = (1 + \epsilon) \sin \theta \sin \Phi \; , \tag{100b}$$

$$\frac{\partial z}{\partial s} = (1 + \epsilon) \cos \theta \; . \tag{100c}$$

For the case of steady equilibrium conditions, the latter set of relations are useful. Either set of relations can be used in the transient problem.

3. Equations of Motion in Natural Coordinates

The vector equation of motion (15) contains the term $\partial \hat{T}/\partial s$ which can be expressed in the form

$$\frac{\partial \hat{T}}{\partial s} = \frac{\partial T}{\partial s} \hat{\tau} + T \frac{\partial \hat{\tau}}{\partial s} \; , \tag{101}$$

where $\partial \hat{\tau}/\partial s$ is the local vector curvature of the line. Using the last of relations (89) we find

$$\frac{\partial \hat{\tau}}{\partial s} = \frac{\partial \theta}{\partial s} \hat{\nu} + \sin \theta \frac{\partial \Phi}{\partial s} \hat{\mu} \; . \tag{102}$$

Also from relations (88) we get

$$\hat{k} = - \sin \theta \; \hat{\nu} + \cos \theta \; \hat{\tau} \; . \tag{103}$$

The residual force $\hat{R}$, given by (18), contains the terms $\partial U_\tau / \partial t$ which can be rewritten in the form

$$\frac{\partial U_\tau}{\partial t} = \frac{\partial \hat{U}}{\partial t} \cdot \hat{\tau} + \hat{U} \cdot \frac{\partial \hat{\tau}}{\partial t} . \tag{104}$$

It will be convenient to denote by $\hat{G}$ that part of $\hat{R}$ which does not depend upon $\partial \hat{\tau}/\partial t$. Accordingly

$$\hat{R} = \hat{G} - m_a \hat{\tau} \, \hat{U} \cdot \frac{\partial \hat{\tau}}{\partial t} \tag{105}$$

and

$$\hat{G} = \lambda (\hat{U}_n - \hat{V}_n) + \Gamma \hat{\tau} \times (\hat{U}_n - \hat{V}_n)$$
$$+ (m_d + m_a) \frac{\partial \hat{U}}{\partial t} - m_a \hat{\tau} (\frac{\partial \hat{U}}{\partial t} \cdot \hat{\tau}) , \tag{106}$$

where

$$\lambda \equiv (\rho/2) \, C_d \, D \, (1 + \epsilon) \, |\hat{U}_n - \hat{V}_n| ,$$

$$\Gamma \equiv \lambda \, \eta \, \cos (\omega_s t + \delta) . \tag{107}$$

Finally we note in analogy to (102) that

$$\frac{\partial \hat{\tau}}{\partial t} = \frac{\partial \theta}{\partial t} \hat{\nu} + \sin \theta \frac{\partial \phi}{\partial t} \hat{\mu} . \tag{108}$$

Consequently using (15), (92) and the above relations of this section we obtain the following component equations of motion

$$(m + m_a) \left\{ \frac{\partial V_\nu}{\partial t} + V_\tau \frac{\partial \theta}{\partial t} - V_\mu \cos \theta \frac{\partial \Phi}{\partial t} \right\}$$

$$- T \frac{\partial \theta}{\partial s} = w \sin \theta + G_\nu , \qquad (109)$$

$$(m + m_a) \left\{ \frac{\partial V_\mu}{\partial t} + (V_\nu \cos \theta + V_\tau \sin \theta) \frac{\partial \Phi}{\partial t} \right\}$$

$$- T \sin \theta \frac{\partial \Phi}{\partial s} = G_\mu , \qquad (110)$$

and

$$m \frac{\partial V_\tau}{\partial t} - [m V_\nu + m_a (V_\nu - U_\nu)] \frac{\partial \theta}{\partial t}$$

$$- [m V_\mu + m_a (V_\mu - U_\mu)] \sin \theta \frac{\partial \Phi}{\partial t}$$

$$- \frac{\partial T}{\partial s} = - w \cos \theta + G_\tau . \qquad (111)$$

The components of $\hat{G}$ in the natural coordinate system are as follows:

$$G_\nu = \lambda (U_\nu - V_\nu) - \Gamma (U_\mu - V_\mu) + (m_d + m_a) \dot{U}_\nu .,$$

$$G_\mu = \lambda (U_\mu - V_\mu) + \Gamma (U_\nu - V_\nu) + (m_d + m_a) \dot{U}_\mu , \quad (112)$$

$$G_\tau = m_d \dot{U}_\tau ,$$

where $\dot{U}_\nu$, $\dot{U}_\mu$, $\dot{U}_\tau$ are the natural coordinate components of the vector $\partial \hat{U} / \partial t$.

It is supposed that the $\hat{U}$ is prescribed in terms of its Cartesian components, U_x, U_y, U_z. Using (88) we find that

$$U_\nu = U_x \cos \theta \cos \Phi + U_y \cos \theta \sin \Phi - U_z \sin \theta$$

$$U_\mu = - U_x \sin \Phi + U_y \cos \Phi , \quad (113)$$

$$U\tau = U_x \sin \theta \cos \Phi + U_y \sin \theta \sin \Phi + U_z \cos \theta.$$

Moreover it is supposed that U_x, U_y, U_z are prescribed functions of x. y. z. and t. Hence they are implicitly related to s, t through the dependent position coordinates of the mooring line. The functions $\dot{U}_\nu$, $\dot{U}_\mu$, and $\dot{U}_\tau$ are related to their Cartesian counterparts $\dot{U}_x$, $\dot{U}_y$ and $\dot{U}_z$ through equations similar to (113).

Finally we note in passing that the evaluation of the coefficient λ requires the relation

$$|\hat{U}_n - \hat{V}_n| = [\ (\ U_\nu - V_\nu)^2 + (U_\mu - V_\mu)^2 \ ]^{1/2} \ . \quad (114)$$

4. Characteristic Form of the Mooring Line Equations
    in Natural Coordinates

In the natural coordinate representation we have the following dependent variables: V_ν, V_μ, V_τ, θ, Φ, T, and ϵ. Each of these is a function of s, t. They are governed by the kinematical relations (93), (94), (95) and by the dynamical relations (109), (110), (111). To these relations, we add the diagnostic equation of state (69) relating T to ϵ plus the two additional variables ξ and ϵ_0, the latter two being governed by the prognostic relations (70) and (71). This gives a total of nine dependent variables and nine equations. We can also add to these the three relations (99a, b, c) or (100a, b, c) governing the three position coordinates x, y, z.

Thus, in summary, the general three-dimensional transient mooring line problem as formulated here (aside from upper end conditions) involves eleven prognostic

relations involving the twelve variables

$$V_\nu, \ V_\mu, \ V_\tau, \ \theta, \ \Phi, \ T, \ \epsilon, \ \xi, \ \epsilon_0, \ x, \ y, \ z$$

plus one diagnostic relation involving T, ϵ, ξ, and ϵ_0 .
Each of the prognostic equations is of quasi-linear form.
For example, referring to (109) to (111) we note that all
derivatives of the dependent variables enter linearly,
their coefficients being functions of the dependent
variables but not of their derivatives. Likewise the
terms G_ν, G_μ, and G_τ are functions of the dependent
variables but not of their derivatives. Any quasi-linear,
hyperbolic system of n equations in n dependent
variables but involving only two independent variables
(s, t in this case) can be recast in a characteristic
form, which facilitates integration of the system (see
for example, Freeman, 1951).

The transformation leading to the characteristic
form of the equations is outlined in Appendix B. The
resulting equations are as follows:

$$\frac{dV_\mu}{dt} + B_1^{\pm} \ \frac{d\Phi}{dt} = G_\mu/(m + m_a) \tag{115a}$$

along

$$\frac{ds}{dt} = \pm \left[T/(m + m_a) \, (1 + \epsilon) \, \right]^{1/2} \, , \tag{115b}$$

where

$$B_1^{\pm} = (V_\nu \cos \theta + V_\tau \sin \theta)$$

$$\mp \, [\, (1 + \epsilon) \, T/(m + m_a) \,]^{1/2} \sin \theta \, ; \tag{115c}$$

$$\frac{dV_\nu}{dt} + B_2^{\pm} \, \frac{d\theta}{dt} = \frac{(G_\nu + w \sin \theta)}{(m + m_a)}$$

$$+ V_\mu \cos \theta \, \frac{d\phi}{dt} \, , \tag{116a}$$

along

$$\frac{ds}{dt} = \pm [\, T/(m + m_a) \, (1 + \epsilon)]^{1/2} \, , \tag{116b}$$

where

$$\dot{B}_2^{\pm} = V_\tau \mp [\, (1 + \epsilon) \, T/(m + m_a)]^{1/2} \, ; \tag{116c}$$

$$\frac{dV_\tau}{dt} \mp \left(\frac{1}{mY}\right)^{1/2} \frac{dT}{dt} = \frac{(G_\tau - w\cos\theta)}{m} + P^\pm$$

$$+ B_3^\pm \frac{dV_\nu}{dt} + B_4^\pm \frac{dV_\mu}{dt} + B_5^\pm \frac{d\theta}{dt} + B_6^\pm \frac{d\phi}{dt} , \qquad (117a)$$

along

$$\frac{ds}{dt} = \pm (Y/m)^{1/2} , \qquad (117b)$$

where

$$Y = K e^{b(\epsilon - \epsilon_0)} \qquad (117c)$$

and $B_3^\pm$, $B_4^\pm$, $B_5^\pm$, $B_6^\pm$, $P^\pm$ are functions to be defined later; finally

$$\frac{d\xi}{dt} = \frac{(\epsilon - \epsilon_0 - \xi)}{\tau_1} , \qquad (118a)$$

along

$$\frac{ds}{dt} = 0 ; \qquad (118b)$$

and

$$\frac{d\epsilon_0}{dt} = S(T, \epsilon_0) , \qquad (119a)$$

along

$$\frac{ds}{dt} = 0 , \qquad (119b)$$

where

$$S(T, \epsilon_0) = \begin{cases} (F(T) - \epsilon_0)/\dot{\tau}_0, & \text{if } \epsilon_0 < F(T) \\ 0 & , \text{if } \epsilon_0 \geq F(T). \end{cases} \qquad (119c)$$

Moreover, from (69)

$$\epsilon = c_0 + \frac{1}{b} \ell n \left(\frac{K + bT + bK_1 \xi}{K} \right), \qquad (120)$$

which implies that Y has the alternative form

$$Y = K + bT + bK_1 \xi . \qquad (121)$$

The coefficients $E_3^{\pm}$ to $B_6^{\pm}$ are given by the following relations:

$$B_3^{\pm} = \pm \frac{m_a \, (m + m_a)}{m Z} \left(\frac{Y}{m}\right)^{1/2} (V_\nu - U_\nu) \, , \qquad (122)$$

$$B_4^{\pm} = \pm \frac{m_a \, (m + m_a)}{m Z} \left(\frac{Y}{m}\right)^{1/2} (V_\mu - U_\mu) \, , \qquad (123)$$

$$B_5^{\pm} = V_\nu - \frac{m_a \, (V_\nu - U_\nu)}{m Z} \lfloor T \mp (m + m_a) \, V_\tau \, (Y/m)^{1/2} \rfloor$$

$$(124)$$

$$B_6^{\pm} = V_\mu \sin \theta - \frac{m_a \, (V_\mu - U_\mu)}{m Z} [\, .T$$

$$\pm \, (m + m_a) \, V_\tau \, (Y/m)^{1/2} \,] \, . \sin \theta$$

$$\pm \, \frac{m_a \, (m + m_a)}{m Z} \, (U_\nu \, V_\mu - U_\mu \, V_\nu) \, (Y/m)^{1/2} \cos \theta \, ,$$

$$(125)$$

where

$$Z = \frac{(m + m_a)}{m} \, (1 + \epsilon) \, Y - T \, . \qquad (126)$$

Moreover, $P^{\pm}$ is given by

$$P^{\pm} = \pm (Y/m)^{1/2} \left\{ \frac{K_1}{Y\tau_1} (\epsilon - \epsilon_0 - \xi) + S(T, \epsilon_0) \right.$$

$$\left. - \frac{m_a}{mZ} [(V_\nu - U_\nu) (w \sin \theta + G_\nu) + (V_\mu - U_\mu) G_\mu] \right\}.$$

$$(127)$$

To these relations we add equations (99a, b, c) for prediction of the Cartesian position coordinates of the line at s,t. An alternative is to employ (100a, b,c) to evaluate these quantities at a given instant from θ and ξ.

Relations (115) and (116) govern the transverse wave modes in the line, while relations (117) govern the longitudinal wave modes, the damping of which are governed by (118) and (119). The nominal signal speeds for the two modes are:

for transverse modes,

$$C' = [T/(m + m_a) (1 + \epsilon)]^{1/2} , \qquad (128)$$

for longitudinal modes,

$$C'' = [Y/m]^{1/2} . \qquad (129)$$

Note, however, that these refer to speeds in the sense of ds/dt. The true speed in the vertical direction can be evaluated from the relation

$$\frac{dz}{dt} = \frac{\partial z}{\partial t} + \frac{\partial z}{\partial s} C, \tag{130}$$

where C is the value given by (128) or (129) depending on the wave mode. Using relation (99c), (100c) and (130) gives

$$\frac{dz}{dt} = (1 + \epsilon) C \cos \theta - V_\nu \sin \theta + V_\tau \cos \theta , \tag{131}$$

along $ds/dt = C$.

The transverse and longitudinal wave modes are coupled through certain non-linear effects. In particular the longitudinal waves are coupled to the transverse waves through the terms involving the time rates of change of V_ν, V_μ, θ and ϕ which appear on the right hand side of (117a). These coupling effects are significant only if $d\theta/dt$ and $d\phi/dt$ are large. Now since

$$\frac{d\theta}{dt} = \frac{\partial \theta}{\partial t} + C \frac{\partial \theta}{\partial s} , \tag{132}$$

where $C = \pm (Y/m)^{1/2}$ for this case, it follows that $d\theta/dt$ can be large if the curvature, $\partial\theta/\partial s$, is large even though the local rate of change of θ may be small. In a similar manner, $d\phi/dt$ can be large if the pitch, $\partial\phi/\partial s$, is large. It is also possible for the angular velocity $\partial\phi/\partial t$ to be large for rotary motion of the line. We note as well that the ν-component, transverse wave modes can be coupled to the μ-component, transverse modes if $d\phi/dt$ is large.

5. Magnitude of the Signal Speeds

The upper limits on C' and C'' are determined respectively by the ultimate values of T and Y. Wilson (1967) gives the following values for 6×37 galvanized steel cable of nominal diameter D_o (inches):

$$m = 4.9 \times 10^{-2} \, D_o^2 \qquad \text{(slugs/ft)}$$

$$T_u = 7.3 \times 10^4 \, D_o^2 \qquad \text{(lbs)}$$

$$Y_u = 6 \times 10^6 \, D_o^2 \qquad \text{(lbs)}$$

$$\varepsilon_u = 0.03 \, .$$

For three strand, regular lay nylon rope:

$$m = 0.9 \times 10^{-2} D_o^2 \qquad \text{(slugs/ft)}$$

$$T_u = 3.1 \times 10^4 D_o^2 \qquad \text{(lbs)}$$

$$Y_u = 8 \times 10^5 D_o^2 \qquad \text{(lbs)}$$

$$\epsilon_u = 0.52 .$$

The latter three values are taken from Paquette and Henderson's data.

The value of m_a is given by

$$m_a = c_a \rho \frac{\pi}{4} D^2 (1 + \epsilon) .$$

If we assume that $D^2 (1 + \epsilon) = D_o^2$ and that $C_a = 1$, we get

$$m_a = 1.1 \times 10^{-2} D_o^2 \quad \text{(slugs/ft)},$$

where D_o is in inches.

Using (128) and (129) with the above data yields the maximum values of the signal speeds as presented in Table 2.

TABLE 2

MAXIMUM SIGNAL SPEEDS

	Steel Cable	Nylon Rope
Transverse (C_u')	1010 ft/sec	1000 ft/sec
Longitudinal (C_u'')	11,000 ft/sec	9400 ft/sec

These speeds are independent of the diameter of the rope

or cable. Also the maximum speeds for nylon rope are

seen to be very close to the corresponding speed for

steel cable. Finally we note that the longitudinal

speed is about a factor of ten larger than the speed

for transverse waves.

6. Characteristic Form of the Mooring Line Equations

    in Cartesian Coordinates

The characteristic relations in natural coordinates

as given in (115), (116), and (117) have a serious potential

pitfall which can lead to difficulties in application.

For those conditions where $\theta = 0$, the solution for ϕ

becomes indeterminate. Note that $B_1^+ = B_1^-$ when $\theta = 0$,

so that only one relation is obtained from (115). Even if

we were to apply these relations only to those cases
where the mean value of θ differs from zero, it is still
possible for the instantaneous value of θ at a particular
point s to be zero due to the presence of transverse
waves.

This difficulty can be circumvented by rewriting
relations (115) to (117) in Cartesian form. The details
of the transformation are lengthy, but straightforward.
This is outlined in Appendix C. In principle we use
relations (88) plus the definitions

$$\alpha = \cos \phi \sin \theta ,$$
$$\beta = \sin \phi \sin \theta , \qquad (132)$$
$$\gamma = \cos \theta .$$

Note that α, β, γ are simply the Cartesian components
of the vector $\hat{\tau}$ (see (88)). Moreover, they must
satisfy the relation

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \qquad (133)$$

and hence only two of these components are independent.

Let V_x, V_y, V_z represent the Cartesian components
of $\hat{V}$. Clearly these are related to x, y, z by

$$\partial x / \partial t = V_x,$$
$$\partial y / \partial t = V_y,$$
$$\partial z / \partial t = V_z .$$

(134)

Also, α, β, γ are related to x, y, z and ϵ by the relations

$$\partial x / \partial s = (1 + \epsilon) \, \alpha ,$$
$$\partial y / \partial s = (1 + \epsilon) \, \beta ,$$
$$\partial z / \partial s = (1 + \epsilon) \, \gamma .$$

(135)

Note that these satisfy (133) automatically since

$$(1 + \epsilon)^2 = (\partial x / \partial s)^2 + (\partial y / \partial s)^2 + (\partial z / \partial s)^2 .$$

(136)

The transformed versions of (115) to (117) in terms of V_x, V_x, V_z, α, β, γ, ϵ, and T can be shown to have the following form:

$$(1 - \alpha^2) \, \frac{dV_x}{dt} - \alpha \beta \, \frac{dV_y}{dt} - \alpha \gamma \, \frac{dV_z}{dt} \mp \left[\frac{(1 + \epsilon)T}{m'} \right]^{1/2} \frac{d\alpha}{dt}$$

$$= (G_{nx} + w \, \alpha \, \gamma) / m'$$

(137a)

along

$$\frac{ds}{dt} = \pm \left[T/m' (1 + \epsilon) \right]^{1/2} ; \tag{137b}$$

$$\beta \frac{dV_x}{dt} + (1 - \beta^2) \frac{dV_y}{dt} - \beta\gamma \frac{dV_z}{dt} \mp \left[\frac{(1 + \epsilon)T}{m'} \right]^{1/2} \frac{d\beta}{dt}$$

$$= (G_{ny} + w \beta v)/m' , \tag{138a}$$

along

$$\frac{ds}{dt} = \pm \left[T/m' (1 + \epsilon) \right]^{1/2} ; \tag{138b}$$

and

$$\alpha \frac{dV_x}{dt} + \beta \frac{dV_y}{dt} + \gamma \frac{dV_z}{dt} \mp (m Y)^{-1/2} \frac{dT}{dt}$$

$$= (G_\tau - w \gamma)/m \pm (Y/m)^{1/2} [K_1 (\epsilon - \epsilon_0 - \xi)/Y\tau_1$$

$$+ S (T, \epsilon_0)] + \delta^{\pm} , \tag{139a}$$

along

$$\frac{ds}{dt} = \pm (Y/m)^{1/2} . \tag{139b}$$

Here we have let

$$m' \equiv m + m_a \, , \qquad\qquad (140)$$

for convenience. The terms G_{nx} and G_{ny} denote the x and y components of the vector $\hat{G} - \hat{\tau} \, G_\tau$. The term $\delta^\pm$ denotes a residual of secondary importance which will hereafter be ignored; it is given in Appendix C and the justification of neglecting it is discussed.

The quantities G_{nx}, G_{ny} and G_τ in terms of Cartesian variables are as follows

$$G_{nx} = \lambda \, [\, (1 - \alpha^2)(U_x - V_x) - \alpha\beta(U_y - V_y) - \alpha\gamma(U_z - V_z)\,]$$

$$+ \, \Gamma \, [\, \beta(U_z - V_z) - \gamma(U_y - V_y)\,]$$

$$+ \, (m_d + m_a) \, [\, (1 - \alpha^2) \, \dot{U}_x - \alpha\beta \, \dot{U}_y - \alpha\gamma \, \dot{U}_z\,] \, , \qquad (141a)$$

$$G_{ny} = \lambda \, [\, - \alpha\beta(U_x - V_x) + (1 - \beta^2)(U_y - V_y) - \beta\gamma(U_z - V_z)\,]$$

$$+ \, \Gamma \, [\, \gamma(U_x - V_x) - \alpha(U_z - V_z)\,]$$

$$+ \, (m_d + m_a) \, [\, - \alpha\beta \, \dot{U}_x + (1 - \beta^2) \, \dot{U}_y - \beta\gamma \, \dot{U}_z\,] \, , \qquad (141b)$$

$$G_\tau = m_d \, (\alpha \, \dot{U}_x + \beta \, \dot{U}_y + \gamma \, \dot{U}_z) \, . \qquad\qquad (141c)$$

In the evaluation of λ the term $|\hat{U}_n - \hat{V}_n|$ now must be determined from the relation

$$|\hat{U}_n - \hat{V}_n| = \left\{ [(1 - \alpha^2)(U_x - V_x) - \alpha\beta(U_y - V_y) - \alpha\gamma(U_z - V_z)]^2 + [-\alpha\beta(U_x - V_x) + (1 - \beta^2)(U_y - V_y) - \beta\gamma(U_z - V_z)]^2 \right\}^{1/2}. \quad (142)$$

Equations (137) to (139) plus (118) and (119) involve the prognostic variables V_x, V_y, V_z α, β, T, ξ, and ϵ_0. All other dependent variables like ϵ, γ, Y are related to these through the appropriate diagnostic relations already stipulated. The Cartesian form of the relations have the advantage that the relations remain determinate when α or β vanish (i.e., $\theta = 0$). Moreover, they do not require the evaluation of trigonometric functions, which can be of economic advantage in sequential numerical integration operations. However, (137) to (139) do have the disadvantage that the time derivatives of V_x, V_y, and V_z enter in all relations. This implies that the numerical integration procedure based on (137) to (139) is more complex than that based on (115) to (117). Clearly this is the price one must pay to be assured of a determinate system under all

conditions. The numerical method of solution based on the
method characteristics for the Cartesian variables is
outlined in the subsequent section.

It will be remarked in passing that relations (137)
to (139) can be derived from the original equations of
motion in Cartesian form by a procedure similar to that
outlined in Appendix B. This procedure in fact was em-
ployed by the writer as a check on the transformation
discussed above, but for a less general case.

7. <u>Numerical Procedure Using the Method of Characteristics</u>

We consider in this section a numerical method of
evaluation of the prognostic variables which is adaptable
to high speed digital computers.

Let $Q(j,k)$ denote any of the dependent variables
at discrete values of s and t defined by

$$s_j = j \, \Delta s. \qquad j = 0, 1, 2. \ . \ . \ .N$$

$$\text{(143)}$$

$$t_k = k \, \Delta t, \qquad k = 0, 1, 2. \ . \ . \ .$$

where $\Delta s = L/N$.

The set of prognostic variables includes V_x, V_y, V_z, α, β, T, ξ, and ϵ_0. It will be supposed that each of these is stipulated for each j at $k = 0$ as initial conditions. The strain ϵ can of course be obtained from (120) and the initial position coordinates x, y, z for each j can be obtained by simple numerical quadrature using (135) with the condition $x = y = z = 0$ at $j = 0$. Alternatively we could supply x, y, z for each j at $k = 0$ and evaluate α, β, ϵ using (135) then get T from (69). The main point is that only eight of the above variables can be specified arbitrarily at time zero.

As end conditions at the sea bed we consider that $x(0,k)$, $y(0,k)$ $z(0,k)$ are zero for all k. For the upper end conditions we will, for the time being, suppose that $V_x(N,k)$, $V_y(N,k)$, and $V_z(N,k)$ are specified for all k as fluctuating values with zero mean value (no net motion over an extended period of time).

The discussion of the computational procedure at a general time step is facilitated by reference to Fig. 16. We assume that V_x, V_y, V_z, α, β, T, ξ, and ϵ_0 have already been evaluated for each j at time level k. We wish to evaluate these same variables at the next time level $k + 1$ for a common value of j (point B). The lines $A_n B$, $n = 1, 2. \ldots .8$, denote the characteristic paths with slope $ds/dt = (-1)^{n+1} c_n$ associated
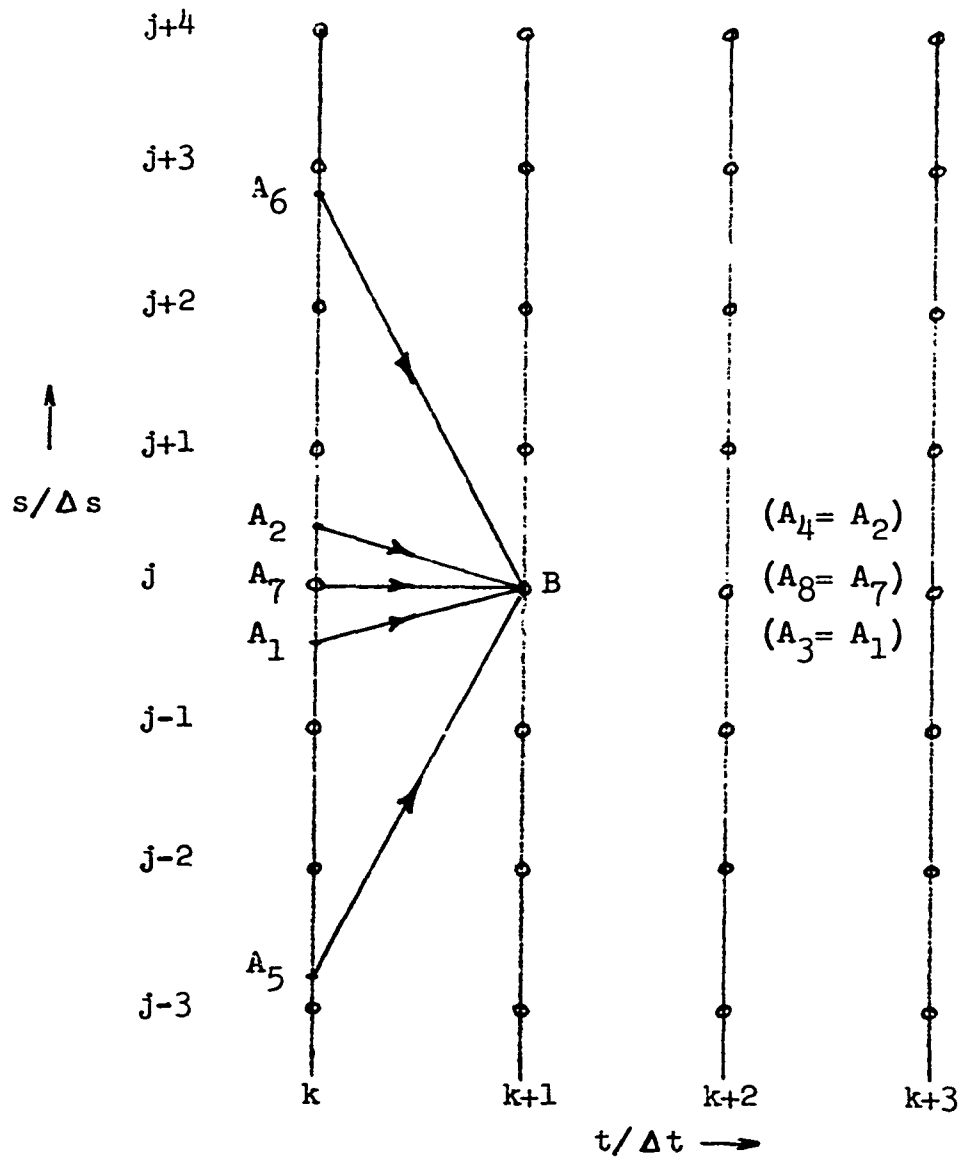
Fig. 16   Schematic of the characteristic paths
          in the discrete s,t grid.

with the eight different prognostic relations obtained
from (137a), (138a), (139a), (118a) and (119a). Specifically

$$C_n = \begin{cases} C_n' & \text{for } n = 1, 2, 3, 4 \\ C_n'' & \text{for } n = 5, 6 \\ 0 & \text{for } n = 7, 8 \end{cases} \qquad (144)$$

where

$$C' = (T/m' (1 + \epsilon))^{1/2} \qquad (145a)$$

$$C'' = (Y/m)^{1/2} , \qquad (145b)$$

$$Y = K + bT + bK_1 \xi . \qquad (145c)$$

The subscripts on C' and C'' imply appropriate average
values of these variables for the path in question, the
first estimates being interpolated values at the points
A_n.

The numerical computation scheme described below
involves a successive approximation procedure similar
to that employed by O'Brien and Reid (1967). This
scheme requires appropriate interpolational relations for
the variables at points A_n. Let a_n represent the values
of $s/\Delta s$ corresponding to points A_n. If the C_n are
appropriate average values then

$$a_n = j + (-1)^n C_n \, \Delta t / \Delta s. \tag{146}$$

For values of j near the end points it is possible for a_n to fall outside the admissible range of j. In this case special procedures are required and are allowed for in the following discussion.

Let J_n be the integer truncation of a_n such that if $a_n > 0$,

$$J_n \leq a_n < J_n + 1. \tag{147}$$

Moreover if

$$0 \leq a_n < N, \tag{148}$$

then a given variable Q can be estimated for points A_n by linear interpolation as follows:

$$Q(a_n, k) \doteq (a_n - J_n) Q(J_n + 1, k)$$

$$+ (J_n + 1 - a_n) Q(J_n, k). \tag{149}$$
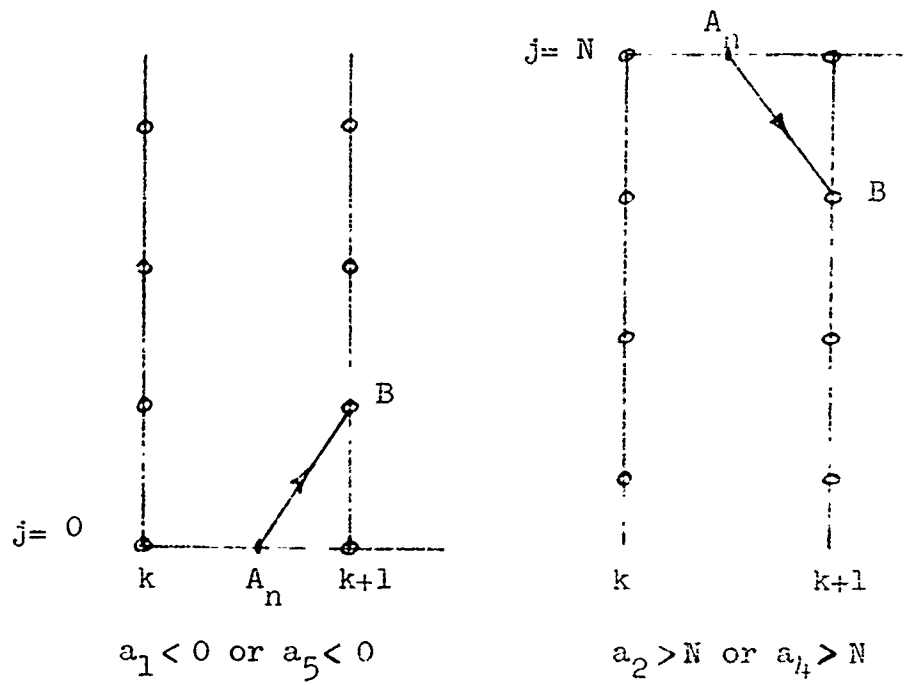
Fig. 17   Schematic illustrating special conditions
           for the characteristic paths near the end
           points of the system.

If (148) is not satisfied, this implies that the characteristic line in question intersects the boundary $j = 0$ or $j = N$ at $k + b_n$ where $0 < b_n < 1$ (see Fig. 17). The value of b_n can be evaluated from

$$b_n = (a_n - J_e)(a_n - j) , \tag{150}$$

where $J_e = 0$ if $a_n < 0$ or $J_e = N$ if $a_n \geq N$. The value of variable Q at point A_n (Fig. 17) can be estimated by the interpolational relation

$$Q(J_e, k + b_n) \doteq b_n Q(J_e, k + 1)$$

$$+ (1 - b_n) Q(J_e, k). \tag{151}$$

Clearly this requires that Q be known at the end points for the time level $k + 1$. Accordingly the calculations of all eight prognostic variables should be carried out for the end points first, making use of the end conditions to replace those relations corresponding to characteristic paths for which a_n falls outside the allowable range.

The discussion of the numerical versions of the prognostic relations is greatly facilitated by employing the following index notation to represent the prognostic variables :

$$Q_m = (V_x, V_y, V_z, \alpha, \beta, T, \xi, \epsilon_0) , \qquad (152)$$

where $m = 1, 2, \ldots .8$. Thus $Q_1 \equiv V_x$, $Q_2 \equiv V_y$ and so on. The prognostic equations all contain terms of the type $B \, d \, Q_m/dt$ which can be approximated by

$$\tfrac{1}{2} [B (j, k + 1) + B (A_n)] [Q_m (j, k + 1) - Q_m (A_n)]/\delta t_n ,$$

where $B(A_n)$ or $Q_m(A_n)$ denotes the value of the variable concerned at point A_n and δt_n denotes the time step appropriate to path n. Specifically for n odd

$$Q_m (A_n) = \begin{cases} Q_m (a_n, k) & \text{if } a_n \geq 0 \\ Q_m (0, k + b_n) & \text{if } a_n < 0 , \end{cases} \qquad (153a)$$

while for n even

$$Q_m (A_n) = \begin{cases} Q_m (a_n, k) & \text{if } a_n < N \\ Q_m (N, k + b_n) & \text{if } a_n \geq N , \end{cases} \qquad (153b)$$

while

$$\delta t_n = \begin{cases} \Delta t & \text{if } 0 \le a_n < N \\[2em] (1 - b_n)\Delta t & \text{if } a_n < 0 , \\[1em] & \text{or } a_n \ge N . \end{cases} \qquad (153c)$$

By employing the above approximations in each of the prognostic equations (137a), (138a), (139a), (118a) and (119a) we obtain eight algebraic relations for the eight unknown Q_m at j, $k + 1$. These can be written in the systematic form

$$\sum_{m=1}^{8} M_{n,m}(j, k) \, Q_m (j, k + 1) = P_n(j,k) , \qquad (154)$$

for $n = 1, 2, \ldots 8$. The matrix of coefficients $M_{n,m}$ and the terms P_n depend upon the variables Q_m at level k and level $k + 1$. These functions are given in Appendix D. We note that the matrix $M_{n,m}$ has the following structure:

	$m = 1$	2	3	4	5	6	7	8
$n = 1$	M_{11}	M_{12}	M_{13}	-1	0	0	0	0
2	M_{21}	M_{22}	M_{23}	1	0	0	0	0
3	M_{31}	M_{32}	M_{33}	0	-1	0	0	0
4	M_{41}	M_{42}	M_{43}	0	1	0	0	0
5	M_{51}	M_{52}	M_{53}	0	0	-1	0	0
6	M_{61}	M_{62}	M_{63}	0	0	1	0	0
7	0	0	0	0	0	0	1	0
8	0	0	0	0	0	0	0	1

Thus (154) can be reduced to the simpler system:

$$\sum_{m=1}^{3} (M_{2n,m} + M_{2n-1,m}) \, Q_m = (P_{2n} + P_{2n-1}) , \qquad (155a)$$

$$Q_{n+3} = (P_{2n} - P_{2n-1}) - \sum_{m=1}^{3} (M_{2n,m} - M_{2n-1,m}) \, Q_m, \qquad (155b)$$

for $n = 1, 2, 3$; and

$$Q_7 = P_7 , \quad Q_8 = P_8 . \qquad (155c)$$

Relation (155a) represents three equations in the three velocity components at j, $k + 1$; this set is easily solved by elimination or by Cramer's rule. Once Q_1, Q_2, Q_3 are obtained, then α, β, T are readily evaluated from (155b) and ξ, ϵ_0 are given by (155c).

In carrying out the calculations it is understood that first estimates of the C_n, P_n, and $M_{n,m}$ are obtained by taking $Q_m(j, k +1)$ on which they depend (Appendix D) as the values $Q_m(j,k)$. Relations (155a, b, c) then give first estimates of $Q_m(j, k + 1)$; the latter are in turn used to re-evaluate the terms, C_n, P_n, and $M_{n,m}$. Repeated use of (155a, b, c,) with the new coefficients, then yields better estimates of the $Q_m(j, k + 1)$. This process is repeated until the difference of successive estimates is less than some acceptable fractional error. Three or four successive approximations usually suffice for sufficiently small time step Δt (O'Brien and Reid, 1967).

At the end points $j = 0$, N we have assumed that the velocity components are stipulated. At $j = 0$ we employ only those prognostic relations for which $a_n \geq 0$. These correspond to the partial set $n = 2, 4, 6, 7, 8$ of relation (154). Since the velocities Q_1, Q_2, Q_3 are given, one can easily solve for the remaining unknowns.

At $j = N$ we must employ the prognostic r lations for which $a_n < N$; these correspond to the pa: _al set $n = 1, 3, 5, 7, 8$. As pointed out previou y, the calculations for $j = 0, N$ should preceed t. se for intermediate points.

When all eight prognostic varia les have been calculated at time level $k + 1$ for all j, then the position coordinates x, y, z for each s at this time level can be evaluated by simple numerical quadrature using the relations

$$
\begin{aligned}
\partial x / \partial s &= (1 + \epsilon)\, \alpha \\
\partial y / \partial s &= (1 + \epsilon)\, \beta \\
\partial z / \partial s &= (1 + \epsilon)\, \gamma \\
\gamma &= (1 - \alpha^2 - \beta^2)^{1/2}
\end{aligned}
\tag{156}
$$

with the conditions $x = y = z = 0$ at $s = 0$.

The whole procedure is then repeated for the next time level. This process is carried out for some pre-selected number of iterations, depending upon the time span of interest and the selected value of Δt. The only limitation on Δt and Δs in this method is that of accuracy and resolution; the method if inherently stable.

Accuracy is improved in principle by selection of small
values of both Δs and Δt. However, economy of computation
plus recognition of accumulation of round-off errors
implies some compromise value for these parameters. It
is recommended in such computations that Δs be taken as
about 100 feet   and that Δt be of the order of 1/10
second.   These values are based upon the considerations
of the possible modes of motion treated in Chapter VI
via linearized perturbation approach.

8. Generalized Upper-End Condition

In the preceding discussion it was assumed that V_x, V_y, and V_z were stipulated as a function of time at the upper end of the line. A more general condition was implied in Chapter II (section 5), in which the dynamics of the surface vessel is taken into account. The purpose of this section is to show how such conditions can be imposed in place of the simple upper conditions of the preceding section.

Let q_i' (i = 1, 2, 3) denote the three Cartesian components (x, y, z respectively) of the anomoly* of translational velocity of the center of mass of the surface vessel, and let Ω_i' denote the Cartesian components of the anomoly* of rotational velocity of the vessel. It will be assumed that the x-axis is taken such as to be parallel to the mean orientation of the longitudinal axis of the vessel. This will correspond to the mean orientation of the surface current for normal mooring with a single line.

Then in the notation of Section 5, Chapter II, we require, for the case of single mooring line, that along $ds/dt = 0$ at $j = N$ (s = L):

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* These are anomolies from the velocities which the vessel would have in the absence of the mooring line.

$$\frac{dR_i'}{dt} = q_i' \quad , \tag{157}$$

$$\frac{d\psi_i'}{dt} = \Omega_i' \quad , \tag{158}$$

$$\frac{dq_i'}{dt} = [- T \, \tau_i / M_i - 2 \, \beta_i \, q_i$$

$$- \sigma_i^2 \, R_i' + T^0 \, \tau_i^0 / M_i] \quad , \tag{159}$$

$$\frac{d\Omega_i'}{dt} = [- J_i / I_i - 2 \, \beta_i' \, \Omega_i'$$

$$- \sigma_i'^2 \, \psi_i' + J_i^0 / I_i] \quad , \tag{160}$$

where $i = 1, 2, 3$; and

$$\hat{V} = \hat{q}' + \hat{\Omega}' \times \hat{p}^0 + \hat{F}(t) \quad . \tag{161}$$

In (159) τ_i are the Cartesian components of the unit vector $\hat{\tau}$ (previously designated α, β, γ).* The term $\hat{F}(t)$ is a prescribed vector function of time denoting the potential velocity of the point of attachment of the mooring line on the vessel, in the absence of the mooring

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* The reader is cautioned that there is a potential ambiguity in the semantics here; the scalars α, β, λ have a meaning quite distinct from the vector components α_i, β_i, λ_i of Section 5, Chapter II.

line. In terms of the notation of Section 5, Chapter II,
$\hat{F}(t) = \hat{\lambda} + \hat{\alpha} \times \hat{p}^0$. The terms J_i are the Cartesian
components of a vector torque which depends upon $\hat{T}, \hat{\tau}$
and the angular displacement of the vessel (see Eq. (31)).

One can readily write finite difference counterparts
of relations (157) to (160) analogous to those for the
prediction of ς. These allow the evaluation of the
vector velocity at the new time level via relation (161).
Having the velocity components at the new time level for
$j = N$ then makes the system equivalent to that described
in the preceding section. It must be noted, however, that
the prediction of q_i' and Ω_i' depend on the new T and
$\hat{\tau}$ at $j = N$. Hence a successive approximation procedure
is required similar to that described in the preceding
section. The initial conditions on the components
R_i', Ψ_i', q_i' and Ω_i' must be prescribed. Bear in mind,
however, that these variables are pertinent only to the
single position $j = N$.

The generalization of the above procedure for the
case of multiple mooring lines is straightforward following
the system indicated in Section 5, Chapter II.

VI. LINEARIZED PERTURBATIONS RELATIVE TO
A COPLANAR EQUILIBRIUM CONFIGURATION

1. Dynamic Equilibrium State

The oceanic current $\hat{U}$ in general consists of:
(1) a quasi-permanent horizontal flow which decreases
slowly with increasing depth; (2) horizontal tidal flow
which is nearly uniform with depth but changes slowly
in direction and speed with time; and (3) rapidly
fluctuating three-dimensional motion associated with
surface gravity waves. The latter phenomena, which has
a spectral band width from about 0.3 to 6 rad/sec, is
confined to a relatively thin surface layer. As a
practical approximation we will suppose that the wave
motion, while directly governing the motion of the vessel,
does not directly influence the mooring line. In effect,
we are implying that the fluctuating part of the current
$\hat{U}$ is confined to a layer comparable to the draft of the
vessel and that over the bulk of the mooring line the
current can be considered steady, at least for durations
of the order of one hour.

On the other hand the line itself is not static but is in a continual fluctuating state of motion in response to the motion of the vessel to which it is secured. Now the drag force is non-linearly related to the relative motion $(\hat{U}_n - \hat{V}_n)$ of the fluid with respect to the line. Accordingly the average* configuration of the line (or lines), for given steady current $\hat{U}$ and a stochastically stationary time sequence of ship motion with given variance, will not in general be the same as the steady state configuration of the line with the same current $\hat{U}$ but with the ship fixed in its mean position (i.e., no wave action). The effect of the fluctuating motion of the line, even though its mean value is zero, is to produce a greater net drag than that due to $\hat{U}$ alone. This can be quite marked if the amplitude of $\hat{V}$ exceeds the magnitude of $\hat{U}$.

Since the average configuration depends upon the perturbation of the line through the non-linear drag force, we will refer to this configuration as the dynamic equilibrium state. It will serve as the reference state in the definition of the perturbation variables of the system. In particular we will denote all reference

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* If the motions of the line are stochastically stationary then time averages are equivalent to expected values in the probability sense (Wiener, 1950).

state variables by a bar over the symbol. This can be interpreted as the average value of that variable at material point s for an ensemble of possible configurations associated with the current $\hat{U}$ and a stipulated ensemble of ship motions.

In regard to the current $\hat{U}$, we will suppose that (below the vessel) this is not only steady but also horizontal and coplanar at all depths. Accordingly, if the statistical distributions of the component velocities of the line have zero mean value and are symmetric about their mean, then we should expect that the dynamic equilibrium configuration will lie in a vertical plane parallel to the current $\hat{U}$. Actually this also assumes that the mean force exerted on the vessel by the combined action of currents, waves, and wind is in the same direction as the current. We will take the x,z-plane to coincide with the plane of this dynamic equilibrium configuration.

Specific relations for $\partial\bar{T}/\partial s$ and $\partial\bar{\theta}/\partial s$ for the mean state are given in the following section in which the perturbation equations are derived.

2. Quasi-Linear Perturbation Equations

The equations of motion expressed in a natural coordinate system based on the mean configuration of the line are the most convenient for analysis of perturbations of the line. These can be derived in a manner similar to that employed in section 3 of Chapter V. In formulating these relations we must be careful to distinguish between an instantaneous vector like $\hat{\tau}$ and its mean value for material point s. Since the overbar notation is combersome for vectors, we will denote the average vector $\hat{\tau}$ as $\hat{\tau}_0$ at given s and similarly for $\hat{\nu}$ and $\hat{\mu}$. The mean unit vectors $\hat{\nu}_0$ and $\hat{\tau}_0$ lie in the x,z-plane while $\hat{\mu}_0$ is parallel to the y-axis.

Let V'_ν, V'_μ, V'_τ hereafter denote the components of $\hat{V}$ in the fixed local coordinate system defined by the vectors $\hat{\nu}_0$, $\hat{\mu}_0$, $\hat{\tau}_0$ at point s. With this understanding the vector acceleration is simply

$$\frac{\partial \hat{V}}{\partial t} = \frac{\partial V'_\nu}{\partial t} \hat{\nu}_0 + \frac{\partial V'_\mu}{\partial t} \hat{\mu}_0 + \frac{\partial V'_\tau}{\partial t} \hat{\tau}_0 \ . \tag{162}$$

The vectors $\hat{\nu}_0$, $\hat{\mu}_0$, $\hat{\tau}_0$ are related to the Cartesian reference vectors by the following relations in analogy to (88) with θ replaced by $\bar{\theta}$ and $\bar{\Phi} = 0$:

$$\hat{\nu}_0 = \hat{i} \cos \overline{\theta} - \hat{k} \sin \overline{\theta} \, ,$$

$$\hat{\mu}_0 = \hat{j} \tag{163}$$

$$\hat{\tau}_0 = \hat{i} \sin \overline{\theta} + \hat{k} \cos \overline{\theta} \, .$$

Now let $\hat{\nu}'$, $\hat{\mu}'$, $\hat{\tau}'$ denote the perturbations of the vectors $\hat{\nu}$, $\hat{\mu}$, $\hat{\tau}$ in sense that $\hat{\nu} = \hat{\nu}_0 + \hat{\nu}'$, etc. It will be understood that the perturbations are small, i.e., $|\hat{\nu}'| \ll 1$, $|\hat{\mu}'| \ll 1$, and $|\hat{\tau}'| \ll 1$. If we also let θ' and Φ' represent associated perturbations of θ and Φ where $|\theta'| \ll 1$ and $|\Phi'| \ll 1$ then we obtain the following approximations based on the use of (89):

$$\hat{\nu}' = \hat{\mu}_0 \cos \overline{\theta} \, \Phi' - \hat{\tau}_0 \, \theta' \, ,$$

$$\hat{\mu}' = - \hat{\nu}_0 \cos \overline{\theta} \, \Phi' - \hat{\tau}_0 \sin \overline{\theta} \, \Phi' \, , \tag{164}$$

$$\hat{\tau}' = \hat{\nu}_0 \, \theta' + \hat{\mu}_0 \sin \overline{\theta} \, \Phi' \, .$$

Now referring to (15) we note that the term $\hat{\tau} \, \partial V_\tau / \partial t$ can be approximated by $\hat{\tau}_0 \, \partial V_\tau' / \partial t$ neglecting products of perturbation quantities. Moreover, referring to (105) we see that the quantity $\hat{U} \cdot \partial \hat{\tau} / \partial t$ is of second

order if $\dot{U}$ is of order $\hat{V}$ or less. Thus (15) reduces approximately to

$$(m + m_a) \frac{\partial \hat{V}}{\partial t} - m_a \dot{\hat{\tau}}_0 \frac{\partial V'_\tau}{\partial t} = \frac{\partial (T \hat{\tau})}{\partial s} - w\hat{k} + \hat{G} . \qquad (165)$$

Now consider the first term on the right-hand side of (165). Letting $T = \bar{T} + T'$ where $T' \ll \bar{T}$ and $\hat{\tau} = \hat{\tau}_0 + \hat{\tau}'$ we obtain after neglecting quantities of second order:

$$\frac{\partial (T \hat{\tau})}{\partial s} \doteq \frac{\partial}{\partial s} (\bar{T} \hat{\tau}_0) + \frac{\partial}{\partial s} (\bar{T} \hat{\tau}') + \frac{\partial}{\partial s} (T' \hat{\tau}_0). \qquad (166)$$

Now from (163) we find that

$$\frac{\partial \hat{v}_0}{\partial s} = - \hat{\tau}_0 \frac{\partial \bar{\theta}}{\partial s} ,$$

$$\frac{\partial \hat{u}_0}{\partial s} = 0 , \qquad (167)$$

$$\frac{\partial \hat{\tau}_0}{\partial s} = \hat{v}_0 \frac{\partial \bar{\theta}}{\partial s} .$$

Hence using (164) and (167), relation (166) takes the form

$$\frac{\partial(T\,\hat{\tau})}{\partial s} \doteq \frac{\partial \overline{T}}{\partial s}\,\hat{\tau}_0 + \overline{T}\,\frac{\partial \overline{\theta}}{\partial s}\,\hat{\nu}_0 + \frac{\partial(\overline{T}\theta')}{\partial s}\,\hat{\nu}_0 - \overline{T}\theta'\,\frac{\partial \overline{\theta}}{\partial s}\,\hat{\tau}_0$$

$$+ \frac{\partial}{\partial s}(\overline{T}\,\sin\overline{\theta}\,\phi')\,\hat{\mu}_0 + \frac{\partial T'}{\partial s}\,\hat{\tau}_0 + T'\,\frac{\partial \overline{\theta}}{\partial s}\,\hat{\nu}_0 \ . \qquad (168)$$

Now using (162), (163), (165), and (168) it follows that the component relations associated with (165) are

$$(m + m_a)\,\frac{\partial V_\nu{}'}{\partial t} = \overline{T}\,\frac{\partial \overline{\theta}}{\partial s} + \frac{\partial(\overline{T}\,\theta')}{\partial s} + T'\,\frac{\partial \overline{\theta}}{\partial s} + w\,\sin\overline{\theta}$$

$$+ \hat{G} \cdot \hat{\nu}_0 \ , \qquad (169a)$$

$$(m + m_a)\,\frac{\partial V_\mu{}'}{\partial t} = \frac{\partial}{\partial s}(\overline{T}\,\sin\overline{\theta}\,\phi') + \hat{G} \cdot \hat{\mu}_0 \ . \qquad (169b)$$

$$m\,\frac{\partial V_\tau{}'}{\partial t} = \frac{\partial \overline{T}}{\partial s} + \frac{\partial T'}{\partial s} - \overline{T}\,\theta'\,\frac{\partial \overline{\theta}}{\partial s} - w\,\cos\overline{\theta} + \hat{G} \cdot \hat{\tau}_0 \qquad (169c)$$

By definition the mean values of the perturbation variables are zero. Hence by averaging (169a, b, c) we get

$$0 = \overline{T}\,\frac{\partial \overline{\theta}}{\partial s} + w\,\sin\overline{\theta} + \overline{\hat{G} \cdot \hat{\nu}_0} \ , \qquad (170a)$$

$$0 = \overline{\hat{G} \cdot \hat{\mu}_0} \ , \qquad (170b)$$

$$0 = \frac{\partial T}{\partial s} - w \cos \bar{\theta} , \tag{170c}$$

the average of $\hat{G} \cdot \hat{\tau}_0$ being zero. These are the relations defining the dynamic equilibrium configuration.

The vector $\hat{G}$ was defined by (106) and (107). We note, first of all, that this simplifies somewhat with $\partial \hat{u}/\partial t = 0$. In order to put the remaining part in a tractable form we will take

$$\hat{U}_n - \hat{V}_n \doteq (U \cos \bar{\theta} - V_\nu') \hat{\nu}_0 + (- V_\mu') \hat{\mu}_0 . \tag{171}$$

Furthermore, the term $\Gamma \hat{\tau} \times (\hat{U}_n - \hat{V}_n)$ will be approximated by $\Gamma \hat{\tau}_0 \times \hat{\nu}_0 \, U \cos \bar{\theta}$. This implies that "strumming" of the line by vortex shedding is allowed only in association with the steady flow U, while that associated with the oscillatory motion of the line is neglected*. Accordingly we have

$$\hat{G} \cdot \hat{\nu}_0 = f[(U \cos \bar{\theta} - V_\nu')^2 + (V_\mu')^2]^{1/2}$$

$$\times (U \cos \bar{\theta} - V_\nu') \tag{172a}$$

_____

*Indeed the phenomenon of vortex shedding is somewhat confused in respect to oscillatory relative flow.

$$\hat{G} \cdot \hat{\mu}_o = f[(U \cos \overline{\theta} - V_\nu')^2 + V_\mu')^2]^{1/2} [- V_\mu$$

$$+ \eta \ U \cos \overline{\theta} \cos (\omega_s \ t + \delta)], \qquad (172b)$$

$$\hat{G} \cdot \hat{\tau}_o = 0, \qquad (172c)$$

where

$$f = (\rho/2) \ C_d \ D \ (1 + \overline{\epsilon}).$$

The latter coefficient can be expressed in the alternative form

$$f = \frac{2}{\pi} \frac{C_d \ \dot{m}_d}{D} \qquad (173)$$

where m_d is the displaced mass of water per unit material length.

The residual equations of motion are obtained by subtracting the corresponding relations in (169) and (170). For convenience we will use the abbreviated notation

$$\beta' \equiv \sin \overline{\theta} \, \Phi' , \tag{174}$$

$$\left. \begin{array}{l} G_\nu' \equiv \hat{G} \cdot \hat{\nu}_o - \overline{\hat{G} \cdot \hat{\nu}_o} , \\[2em] G_\mu' \equiv \hat{G} \cdot \hat{\mu}_o . \end{array} \right\} \tag{175}$$

The term β' in fact represents the perturbation of the y component of $\hat{\tau}$ as defined by (132). The resulting dynamic perturbation relations are:

$$m' \frac{\partial V_\nu'}{\partial t} = \frac{\partial \overline{T}}{\partial s} \theta' + T' \frac{\partial \overline{\theta}}{\partial s} + G_\nu' , \tag{176a}$$

$$m' \frac{\partial V_\mu'}{\partial t} = \frac{\partial \overline{T}}{\partial s} \beta' + G_\mu' , \tag{176b}$$

$$m \frac{\partial V_\tau'}{\partial t} = \frac{\partial T'}{\partial s} - \theta' \overline{T} \frac{\partial \overline{\theta}}{\partial s} . \tag{176c}$$

where $m' = m + m_a$. The relations (176a, b) are only quasi-linear since the terms G_ν' and G_μ' are, in general, non-linear functions of the normal components of velocity of the line. All other terms are linear functions of the perturbation variables.

To these relations we add the following linearized versions of the kinematical relations (93) to (95):

$$(1 + \overline{\epsilon}) \frac{\partial \theta'}{\partial t} = \frac{\partial V_\nu'}{\partial s} + V_\tau' \frac{\partial \overline{\theta}}{\partial s} \tag{177a}$$

$$(1 + \overline{\epsilon}) \frac{\partial \beta'}{\partial t} = \frac{\partial V_\mu'}{\partial s} \tag{177b}$$

$$\frac{\partial \epsilon'}{\partial t} = \frac{\partial V_\tau'}{\partial s} - V_\nu' \frac{\partial \overline{\theta}}{\partial s} \quad . \tag{177c}$$

Finally the linearized versions of the load-strain relations (69) to (71) are taken as follows

$$\tau_1 \frac{\partial \xi'}{\partial t} = (\epsilon' - \xi') , \tag{178}$$

$$T' = Y_0 \epsilon' - K_1 \xi' , \tag{179}$$

where ϵ_0 is regarded as a constant $\left(\text{the maximum tension being assumed such that } \epsilon_0 > F\left(T_{max}\right)\right)$. In the above relations $\epsilon' = \epsilon - \overline{\epsilon}$, $\xi' = \xi - \overline{\xi}$ where $\overline{\xi} = \overline{\epsilon} - \epsilon_0$ and

$$Y_0 = K + b \overline{T} + b K_1 \overline{\xi} , \tag{180}$$

which represents the dynamic spring coefficient for the mean conditions at s.

3. Associated Perturbation Energy Equation

If we multiply (176a, b, c) by V_ν', V_μ', and V_T' respectively, then multiply (177a, b, c) by $\overline{T}\theta'$, $\overline{T}\beta'$, and T', then multiply (178) by $-K_1(\epsilon' - \xi')/\tau_1$ and add the resulting relations we find the following energy relation after using (179) and collecting terms:

$$\frac{\partial}{\partial t}\left\{\left[\frac{1}{2}m'\left(V_\nu'^2 + V_\mu'^2\right) + \frac{1}{2}m\,V_T'^2\right]\right.$$

$$+ \left[\frac{1}{2}Y_0\,\epsilon'^2 - K_1\,\epsilon'\,\xi' + \frac{1}{2}K_1\,\xi'^2\right]$$

$$+ \left[\frac{1}{2}(1 + \overline{\epsilon})\,\overline{T}\,(\theta'^2 + \beta'^2)\right]\right\}$$

$$= \frac{\partial}{\partial s}\left[\,\overline{T}\,(\theta'\,V_\nu' + \beta'\,V_\mu') + T'\,V_T'\right]$$

$$+ G_\nu'\,V_\nu' + G_\mu'\,V_\mu' - \frac{K_1}{\tau_1}(\epsilon' - \xi')^2\,. \tag{181}$$

Comparing this energy relation with (85), it appears that we have introduced a new form of energy term involving $(\theta'^2 + \beta'^2)$. Actually the last two sets of terms in brackets on the left side of (181) represent dynamic elastic energy associated with the perturbation. Only part of the anomaly of elongational strain is represented in the term ϵ'; it is that part related to motion

parallel to the reference configuration. If transverse
waves exist relative to this reference, then second order
elongational strain occurs which is proportional to
$(\theta'^2 + \beta'^2)$. Thus the term $(1/2)(1 + \bar{\epsilon})\,\overline{T}\,(\theta'^2 + \beta'^2)$
is the elastic energy associated with transverse perturbation,
while the term

$$\left[\frac{1}{2}\,Y_0\,\epsilon'^2 - K_1\,\epsilon'\,\zeta' + \frac{1}{2}\,K_1\,\zeta'^2\right]$$

is the elastic energy associated with the longitudinal
perturbations. Note that this will always be positive
as long as $Y_0 > K_1$ which is certainly the case (see
Chapter III).

We note that terms involving the curvature of the
mean configuration $(\partial\bar{\theta}/\partial s)$ which were present in (176a, c)
(177a, c) have no contribution in the above perturbation
energy equation. These terms lead to coupling of, and
hence energy exchange between, the transverse and
longitudinal perturbations but do not account for any
loss or gain for the total dynamic energy budget. On
the other hand, the last three terms involving drag effect
and hysteresis indeed do significantly influence the
total energy budget.

If the perturbations represent a stochastically
stationary process then the average total energy is
constant in time (or at least only slowly varying)
and hence the average of the terms on the right side
of (181) should produce a balance. The drag terms are
particularly significant in this balance of energy loss
and energy gain per unit time. This will be considered
in a statistical sense in the subsequent section.

4. Statistical Considerations of the Drag Force

    In order to arrive at a rational approximation
of the highly non-linear relations for the drag force,
we will consider certain critical statistics associated
with this force for a simple, yet realistic, stochastic
model of the line motion. The motions of the sea
surface and of the surface vessel, in response to the
surface waves, can be considered to represent a Gaussian
process in the first approximation (St. Denis and Pierson,
1955). We will suppose, moreover, that V_ν' and V_μ'
are also of Gaussian character. The principal additional
assumptions are that these variables are statistically
independent and that they have the same variance σ^2
and zero mean value. In general σ^2 will vary with
position s on the line. The assumption of equal

variances is perhaps the most stringent of the assumptions, but considerably simplifies the analysis of the statistics related to the drag force.

For simplicity of notation in the present analysis we will let

$$U \cos \bar{\theta} \equiv \bar{x},$$

$$U \cos \bar{\theta} - V_\nu' \equiv x, \qquad (182)$$

$$V_\mu' \equiv y .$$

Since V_ν' is assumed to be Gaussian with zero mean and variance σ^2, then x is Gaussian with mean $\bar{x}$ and variance σ^2. Moreover the joint probability density of x and y, in view of the foregoing assumptions is

$$F(x, y) = \frac{1}{2\pi \sigma^2} \epsilon^{-[(x - \bar{x})^2 + y^2]/ 2 \sigma^2} . \qquad (183)$$

In the above notation, the component of $\hat{G}$ in the direction $\hat{\nu}_0$ is

$$\hat{G} \cdot \hat{\nu}_0 = f (x^2 + y^2)^{1/2} x . \qquad (184)$$

Its mean value is given by *

$$\overline{\hat{G} \cdot \hat{\nu}_0} = \frac{f}{2\pi\,\sigma^2} \iint\limits_{-\infty}^{\infty} (x^2 + y^2)^{1/2} \; x \; e^{-[(x - \overline{x})^2 + y^2]/2\,\sigma^2} \; dx \; dy. \qquad (185)$$

By making the transformation $x = r \cos \alpha$ and $y = r \sin \alpha$, the integral is converted to the form

$$\overline{\hat{G} \cdot \hat{\nu}_0} = \frac{f}{2\pi\,\sigma^2} \int\limits_{0}^{\infty} \int\limits_{0}^{2\pi} r^3 \cos\alpha \; e^{-(r^2 + \overline{x}^2 - 2\,\overline{x}\,r \cos\alpha)/2\,\sigma^2} \; dx \; dr. \qquad (186)$$

This integral can be evaluated in terms of known functions. The result is

$$\overline{\hat{G} \cdot \hat{\nu}_0} = f \frac{\sigma^2}{2} \sqrt{\frac{\pi}{2}} \; \zeta \; e^{-\zeta^2/4} \left\{ (\zeta^2 + 3) \, I_0(\zeta^2/4) + (\zeta^2 + 1) \, I_1(\zeta^2/4) \right\}, \qquad (187)$$

where

---

* We are ignoring any variation of drag coefficient with velocity in this analysis.

$$\zeta \equiv \overline{x}/\sigma = U \cos \overline{\theta}/\sigma ,$$

$$\sigma \equiv \sqrt{\overline{V_\nu'^2}} = \sqrt{\overline{V_\mu'^2}} ,$$

(188)

and $I_0(z)$, $I_1(z)$ are modified Bessel functions of order indicated. From the properties of the latter functions it is readily shown that the mean drag force has the following limiting forms

$$\overline{\hat{G} \cdot \hat{\nu}_0} = \begin{cases} f \ (U \cos \overline{\theta})^2, & \text{for} \ U \cos \overline{\theta} \gg \sigma \\ \dfrac{3}{2} \sqrt{\dfrac{\pi}{2}} \ f \ \sigma \ U \cos \overline{\theta}, & \text{for} \ U \cos \overline{\theta} \ll \sigma . \end{cases}$$

(189)

In the latter case, it is evident that the standard deviation (σ) of V_ν' or V_μ' has a very definite control on the curvature of the equilibrium configuation through relation (170a).

Another quantity of considerable concern is the mean rate of energy dissipation by the drag force. This is given by

$$\mathbb{D} \equiv f[\overline{(U \cos \overline{\theta} - V_\nu')^2 + (V_\mu')^2]^{1/2}[(V_\nu')^2 + (V_\mu')^2]} .$$

(190)

In terms of the notation (182) this can be written in the form

$$\mathbb{D} = f \overline{r[(x - \overline{x})^2 + y^2]}$$

$$= f(\overline{r^3} + \overline{x}^2 \, \overline{r} - 2\overline{x} \, \overline{rx}), \tag{191}$$

where $r = (x^2 + y^2)^{1/2}$. Again using (183) to evaluate the averages in the above relation we find

$$\mathbb{D} = f \sigma^3 \sqrt{\frac{\pi}{2}} \, e^{-\zeta^2/4} \, [\zeta^2 + 3) \, I_0 (\zeta^2/4)$$

$$+ \zeta^2 \, I_1 (\zeta^2/4)], \tag{192}$$

where ζ is as defined in (188).

We can now define an effective dynamic drag coefficient $f \, v$ such that

$$\mathbb{D} = f \, v \, \overline{(V_\nu'^2 + V_\mu'^2)}. \tag{193}$$

Now $\overline{V_\nu'^2} = \overline{V_\mu'^2} = \sigma^2$ in the foregoing analysis and hence

$$v = \frac{\sigma}{2} \sqrt{\frac{\pi}{2}} e^{-\zeta^2/4} [(\zeta^2 + 3) I_0(\zeta^2/4)$$

$$+ \zeta^2 I_1(\zeta^2/4)]. \tag{194}$$

This has the limiting values

$$v \doteq \begin{cases} U \cos \overline{\theta}, & \text{for } U \cos \overline{\theta} \gg \sigma \\ \\ \frac{3}{2} \sqrt{\frac{\pi}{2}} \sigma, & \text{for } U \cos \overline{\theta} \ll \sigma. \end{cases} \tag{195}$$

A plot of v/σ versus ζ is shown in Fig. 18. This indicates a very rapid transition from one limiting form to the other, centered at $\zeta = \frac{3}{2} \sqrt{\frac{\pi}{2}} \doteq 1.88$ (i.e., $U \cos \overline{\theta} = 1.88 \sigma$). The behavior of the function $\overline{\hat{G} \cdot \hat{\nu}_0}/f \sigma^2 \zeta$ is indistinguishable from v/σ for all practical purposes.

Based upon the above analysis we will adopt the following approximation of the components of $\hat{G}$:

$$\hat{G} \cdot \hat{\nu}_0 \doteq f v (U \cos \overline{\theta} - V_\nu') , \tag{196a}$$

$$\hat{G} \cdot \hat{\mu}_0 \doteq - f v V_\mu' + \eta f v U \cos \overline{\theta} \cos (\omega_s t + \delta) . \tag{196b}$$
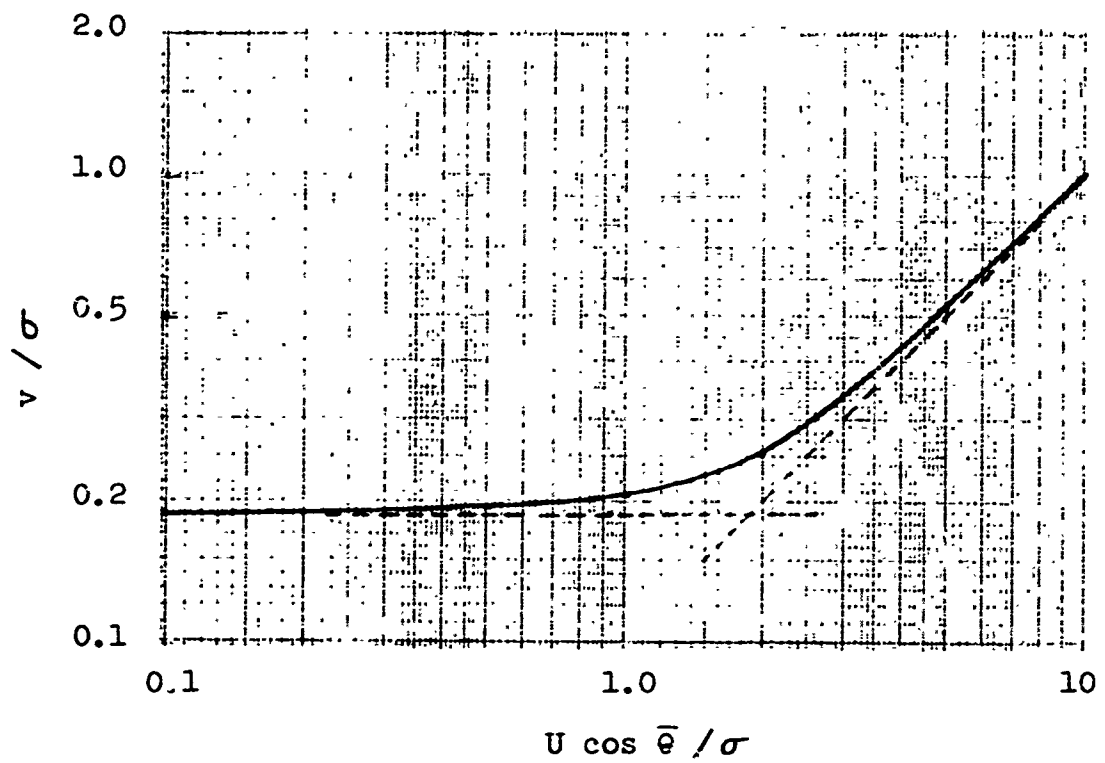
Fig. 18 v/σ versus $\dot{U} \cos \overline{\theta}/\sigma$ based upon a
Gaussian stochastic model for drag force.

Thus

$$G_\nu' = - f v V_\nu' \tag{197}$$

where v is at most a function of s. This approximation
with v as given by (194) leads to the appropriate mean
drag and also the appropriate rate of energy dissipation
by drag force.

5. Perturbations of a Nylon Mooring Line about a Straight Equilibrium Configuration

In the case of nylon line, the net weight per unit
length in water is negligible and can be ignored for all
practical purposes. From (170c) this implies that $\overline{T}$
is essentially uniform over the whole line. Now from
(170a) we find that the curvature of the equilibrium
configuration is given by the mean drag force per unit
length divided by $\overline{T}$. If the latter is large, the mean
configuration will tend to be nearly straight. In this
limiting case of negligible $\partial\overline{\theta}/\partial s$, there is no coupling
between transverse and longitudinal oscillations of the
line. Hence, these modes of oscillation can be analysed
independently. We will consider this situation in the
present section.

In the considerations of the transverse oscillations we will examine two limiting cases in respect to the drag force. These are the cases where $U \cos \bar{\theta} > 5 \sigma$ and $U \cos \bar{\theta} < 0.5 \sigma$ (see Fig. 18). In the first case the term v is independent of σ and the problem is entirely linear. In the second case v is directly proportional to σ which in turn is a function of s dependent upon the statistical behavior of the solution for V_{ν}'. Clearly the latter case is a non-linear problem which must be treated quite differently from the first.

Case A. Transverse Modes $(U \cos \bar{\theta} > 5\sigma)$: In this case $v \doteq U \cos \bar{\theta}$, which will be treated as a constant, at least over that portion of the line where the transverse oscillations are significant. For realistic values of f v, we will find that the transverse modes are not significant at depths greater than about 1000 ft below the vessel and hence the approximation of uniform v seems justifiable in respect to the analysis of these perturbations.

With uniform $\bar{T}$ and zero mean curvature, relations (176a) and (177a) reduce to

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$$m' \frac{\partial V_\nu'}{\partial t} \doteq \overline{T} \frac{\partial \theta'}{\partial s} - f \, v \, V_\nu' \, , \tag{198}$$

$$(1 + \overline{\epsilon}) \frac{\partial \theta'}{\partial t} \doteq \frac{\partial V_\nu}{\partial s} \, . \tag{199}$$

Eliminating θ' between these relations yields

$$\frac{\partial^2 V_\nu'}{\partial t^2} = C_1^2 \frac{\partial^2 V_\nu'}{\partial s} - \frac{f \, v}{m'} \frac{\partial V_\nu'}{\partial t} \, , \tag{200}$$

where

$$C_1 \equiv (\overline{T}/m' \, (1 + \overline{\epsilon}))^{1/2} \, . \tag{201}$$

This has elementary solutions of the form

$$V_\nu' = A \, e^{i(\omega t + ks)} \, , \tag{202}$$

where

$$k^2 = (\omega^2 - i \, f \, v \, \omega/m')/C_1^2 \, . \tag{203}$$

If ω is regarded as real then k has the roots

$$k = \pm (\varkappa - i\,\mu) \tag{204}$$

where

$$\varkappa = |k| \cos \alpha/2$$

$$\mu = |k| \sin \alpha/2 \tag{205}$$

in which

$$|k| = \omega[\, 1 + (f\ v/m'\ \omega)^2\,]^{1/4}/C_1 \tag{206}$$

and

$$\alpha = \tan^{-1} (f\ v/m'\ \omega)\ . \tag{207}$$

Note that $\varkappa$ is an odd function of ω while μ is even.

A more general solution is obtained by forming a linear combination of solutions of the type (202) for the two admissible values of k for given ω. Such a solution satisfying the condition $V_\nu' = 0$ at $s = 0$ for all t is

$$V_\nu' = A [e^{\mu s} \cos (\omega t + \varkappa s + p)$$

$$- e^{-\mu s} \cos (\omega t - \varkappa s + p)] , \qquad (208)$$

in which we have allowed for an arbitrary phase angle p.
The first term in the brackets represents a progressive
wave traveling downwards along the mooring line, its
amplitude decreasing exponentially in the direction of
travel. The second term is a reflected wave traveling
upwards along the line and attenuated exponentially in the
direction of travel. If the attenuation is sufficiently
large, or the line sufficiently long, then the reflected
wave may not even exist. The amplitude of V_ν' at
the position of the vessel $(s = L)$ can be shown to be
given by

$$A (2 \cosh 2 \mu L - 2 \cos 2 \varkappa L)^{1/2} . \qquad (209)$$

If $\mu L \gg 1$ then this reduces simply to $A e^{\mu L}$.

Using (173) we find that

$$\frac{f \; v}{m' \omega} \;=\; \left(\frac{2}{\pi} \; \frac{C_d \; m_d}{m'} \right) \; \frac{v}{\omega \, D} \; . \tag{210}$$

For nylon line, the factor in parentheses is approximately $1/\pi$. Suppose v is only 1 ft/sec and $\omega = 2\pi/10$ rad/sec, corresponding to a 10 second period excitation. Then for a 2 inch diameter line

$$\frac{f \; v}{m' \omega} \;\doteq\; 3 \; .$$

Using relations (205) to (207) and taking $C_1 = 700$ ft/sec (corresponding to $\overline{T}$ equal to about half of the ultimate strength) yields

$$\varkappa \;\doteq\; 1.3 \times 10^{-3}/\text{ft}$$

$$\mu \;\doteq\; 0.9 \times 10^{-3}/\text{ft} \; .$$

The wave length is $2\pi/\varkappa$ or about 4800 ft; on the other hand an e-fold attentuation of amplitude occurs in an arc length of $1/\mu$ or about 1100 ft. The attenuation of

amplitude in one wave length is about 99 per cent. Thus
for all practical purposes, the transverse waves are
critically damped in one wave length.

The above example serves to demonstrate the extreme
importance of drag effect on the transverse oscillations.
Typical values of $fv/m'|\omega|$ exceed unity. In the
limiting case of very large values of this parameter we
find

$$|\varkappa| \sim |\mu| \sim \left\{ \frac{|\omega| \; f \; v \; (1 + \bar{\epsilon})}{\bar{T}} \right\}^{1/2} , \qquad (211)$$

for $f \; v/m' \; |\omega| \gg 1$. This situation corresponds to
negligible inertial effect (the term $m' \; \partial V_\nu'/\partial t$ being
negligible in (198)).

The solution for V_ν' given by (208) is valid
provided that the excitation at the upper end is periodic
in time (with period $2\pi/\omega$). A more realistic condition
at the upper end would allow for the representation of
V_ν' as a general time sequence with stipulated statistical
properties. We will employ here the Fourier-Stieltjes
integral representation as follows:

$$V_\nu' \; (L,t) \; = \; \int_{-\infty}^{\infty} e^{i\omega t} \; dF_1 \; (\omega) \; , \qquad (212)$$

where $dF_1(\omega)$ is a stochastic complex variable having the following statistical property

$$\overline{dF_1{}^*(\omega)\ dF_1(\omega')} = \delta'(\omega - \omega')\ \psi_1(\omega)\ d\omega\ , \tag{213}$$

where $\delta'(x)$ is the Dirac delta function, $dF_1{}^*$ is the complex conjugate of dF_1, and $\psi_1(\omega)$ is the variance spectrum associated with $V_\nu'(L, t)$. The variance of V_ν' at the upper end of the line is

$$\sigma_1{}^2(L) = \int_{-\infty}^{\infty} \psi_1(\omega)\ d\omega\ . \tag{214}$$

The function $dF_1(\omega)$ also has the following symmetry property

$$dF_1(-\omega) = dF_1{}^*(\omega)\ , \tag{215}$$

which assures that the representation (212) leads to real values of V_ν'.

For other positions on the line, the generalization of (202), taking into account that the waves are essentially attenuated completely before they reach the anchor ($\mu L \gg 1$), is as follows

$$V_\nu'(s,t) = \int_{-\infty}^{\infty} e^{\mu(s - L)} e^{i(\omega t + \varkappa s)} dF_1(\omega) . \qquad (216)$$

This reduces to (212) at $s = L$. From (216) and (213) we find that the variance of V_ν' at any position s is given by

$$\sigma_1{}^2(s) = \int_{-\infty}^{\infty} e^{2\mu(s - L)} \psi_1(\omega) d\omega , \qquad (217)$$

where μ is nearly proportional to $|\omega|^{1/2}$, see (211).

In the case of transverse perturbations in the lateral (y) direction, the appropriate equations (under the conditions of constant $\overline{T}$) are

$$m' \frac{\partial V_\mu'}{\partial t} = \overline{T} \frac{\partial \beta'}{\partial s} - f v V_\mu' + f v P \qquad (218)$$

$$(1 + \overline{\epsilon}) \frac{\partial \beta'}{\partial t} = \frac{\partial V_\mu'}{\partial s} , \qquad (219)$$

where P represents a random function associated with the vortex shedding. We can represent this as follows

$$P = \int_{-\infty}^{\infty} e^{i\omega t} \, dQ(\omega) \quad , \tag{220}$$

where

$$\overline{dQ*(\omega) \; dQ(\omega')} = \delta'(\omega - \omega') \; \psi_S(\omega) \; d\omega \tag{221}$$

and

$$\int_{-\infty}^{\infty} \psi_S(\omega) \; d\omega = (\eta \; U \cos \overline{\theta})^2 / 2 \; . \tag{222}$$

If the vortex shedding were exactly periodic then its spectral function would be

$$\psi_S(\omega) = \delta'(|\omega| - \omega_S) (\eta \; U \cos \overline{\theta})^2 / 4 \; . \tag{223}$$

However, the above representation for the vortex shedding will allow some smearing over a finite band width.

Eliminating β' between (218) and (219) and using (220) yields the non-homogeneous relation

$$\frac{\partial^2 V_\mu{'}}{\partial t^2} - C_1{}^2 \frac{\partial^2 V_\mu{'}}{\partial s^2} + \frac{fv}{m'} \frac{\partial V_\mu{'}}{\partial t} =$$

$$fv \int_{-\infty}^{\infty} i \, \omega \, e^{i\omega t} \, d \, Q(\omega) \, . \tag{224}$$

An appropriate solution is

$$V_\mu{'}(s,t) = \int_{-\infty}^{\infty} e^{\mu(s - L)} \, e^{i (\omega t + \varkappa s)} \, dF_2(\omega)$$

$$+ \int_{-\infty}^{\infty} (1 + i \, m'\omega/fv)^{-1} [1 - e^{\mu(s - L) + i\varkappa s}$$

$$-e^{-\mu s - i\varkappa s}] \, e^{i\omega t} \, dQ(\omega) \, , \tag{225}$$

where

$$\overline{dF_2{*}(\omega) \, dF_2(\omega')} = \delta(\omega - \omega') \, \psi_2(\omega) \, d\omega \, . \tag{226}$$

Moreover, dF_2 and dQ are regarded as statistically independent for all ω; this assumes that the input by the surface vessel is completely indepdent of the vortex shedding and vice versa. Considering that $\mu L \gg 1$, the solution (225) reduces essentially to zero at $s = 0$ while at the upper end we have

$$V_\mu{}'(L,t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dF_2(\omega), \qquad (227)$$

which is governed by the motion of the vessel.

The variance of $V_\mu{}'$ at any s is given approximately by the relation

$$\sigma_2{}^2(s) = \int_{-\infty}^{\infty} e^{2\mu(s-L)}\, \psi_2(\omega)\, d\omega$$

$$+ \int_{-\infty}^{\infty} [1 + (m'\,\omega/fv)^2]^{-1} \Big\{ 1 - 2[(e^{\mu(s-L)} + e^{-\mu s}]\cos \varkappa s + e^{2\mu(s-L)}$$

$$+ e^{-2\mu s}\Big\}\, \psi_s(\omega)\, d\omega. \qquad (228)$$

Again considering that $\mu L \gg 1$, the above relation gives essentially zero value at $s = 0$ and at the upper end we get simply

$$\sigma_2^2 (L) = \int_{-\infty}^{\infty} \psi_2 (\omega) \, d\omega .$$
(229)

Case B. Transverse Modes $(U \cos \overline{\theta} < 0.5\sigma)$:

Relations (200) and (224) are still valid for this case, however, we now have

$$v \doteq \frac{3}{2} \sqrt{\frac{\pi}{2}} \, \sigma(s)$$
(230)

at least under the condition that the variance of V_ν' and V_μ' have the common value σ at position s. We have seen in the previous case that for intermediate and large depths below the vessel, the oscillations in the lateral direction may actually dominate. Hence the present analysis is limited to the upper reaches of the line where σ_1^2 remains comparable to σ_2^2, assuming that these are nearly equal at the vessel.

As before we will employ the representation (212) fcr the input sequence $V_\nu(L, t)$. For other s we then propose a solution of the form

$$V_\nu'(s,t) = \int_{-\infty}^{\infty} M(s, \omega) e^{i(\omega t + N(s,\omega)} dF_1(\omega), \quad (231)$$

where functions $M(s,\omega)$ and $N(s,\omega)$ remain to be determined. In order for this form to be consistent with (212) we require that

$$M(L, \omega) = 1, \quad (232a)$$

$$N(L, \omega) = 0. \quad (232b)$$

It will be observed that the solution (216) for case A is simply a special case of (231) in which M is an exponential function of s and the phase function N is linear in s.

Using (231) and the statistical property (213) we find that the variance of V_ν' is given by

$$\sigma_1{}^2(s) = \int_{-\infty}^{\infty} M^2(s, \omega) \psi_1(\omega) d\omega, \quad (233)$$

where the spectral function $\psi_1 (\omega)$ is presumed to be stipulated.

Relation (231) will satisfy (200) provided that

$$- \omega^2 M = C_1^2 [M'' + i\, 2\, M'\, N' + i\, M\, N'' - M\, (N')^2] - i\, f\, v\, \omega\, M/m' , \tag{234}$$

where M', N' imply derivatives with respect to s. The functions M and N are considered real. Hence upon separating real and imaginary parts of (234) we obtain the following two relations:

$$N' = \pm \left\{ \left(\frac{\omega}{C_1}\right)^2 + \frac{M''}{M} \right\}^{1/2} \tag{235}$$

and

$$(M^2\, N')' = M^2\, f\, v\, \omega/m'\, C_1^2 . \tag{236}$$

Moreover from (230) and (233) we have

$$v = \frac{3}{2} \sqrt{\frac{\pi}{2}} \left\{ \int_{-\infty}^{\infty} M(s,\omega)\, \psi_1 (\omega)\, d\omega \right\}^{1/2} . \tag{237}$$

The set of equations (235) to (237) represents a highly non-linear, integro-differential system for the unknown functions M and N. An exact general solution is not known. However it is possible to obtain approximate solutions for two limiting cases. These are based upon the magnitude of the non-dimensional parameter $fv/\omega m'$.

For $fv/\omega m' \ll 1$, then inertia dominates over drag effect and we should expect that M will be a slowly varying function of s. This implies that $(\omega/C_1)^2$ dominates over the term M''/M in relation (235). Thus $N' \doteq \pm \omega/C_1$ as a limiting case. Using this in (236) leads to

$$(M^2)' \doteq \pm M^2 \, f \, v \, /m' \, C_1 \, , \tag{238}$$

which does not involve ω explicitly. Thus M is independent of ω for this case and consequently (237) and (214) yield

$$v = \frac{3}{2} \sqrt{\frac{\pi}{2}} \ \sigma \ (L) \ M(s). \tag{239}$$

Using (238) and (239) leads to the differential equation

$$M^{-3} \ (M^2)' = \frac{3}{2} \sqrt{\frac{\pi}{2}} \ f \ \sigma(L)/m'C_1 \, , \tag{240}$$

in which our choice of sign confines attention to waves propagating downwards along the line. This assumes that the line is sufficiently long that the transverse modes are completely damped before reaching the anchor point (a rather stringent assumption for this case). A solution of (240) satisfying condition (232a) is

$$M = [1 + \hbar_1 (L - s)]^{-1} , \tag{241}$$

where

$$\hbar_1 \equiv \frac{3}{4} \sqrt{\frac{\pi}{2}} \; f \; \sigma(L)/m'C_1 . \tag{242}$$

Finally, from (233), (214), and the above result we get for any $\psi_1 (\omega)$:

$$\sigma^2(s) = [1 + \hbar_1 (L - s)]^{-2} \sigma^2(L). \tag{243}$$

Note that the attenuation coefficient $\hbar_1$ is proportional to $\sigma(L)$ according to (242).

For the case $f \, v/\omega \, m' \gg 1$, then drag effect dominates over inertia and hence we expect in this case that M''/M greatly exceeds $(\omega/C_1)^2$. This is the normal situation which would be expected for nylon line. This implies from (235) that

$$N' \doteq (M''/M)^{1/2} , \qquad (244)$$

where our attention is again confined to waves propagating downwards along the line (and attenuated before reaching the anchor point). As a generalization of the solution (241) we will consider in this case that

$$M = [1 + k_2 (L - s)]^{-n} , \qquad (245)$$

where k_2 and n are to be determined for this case. Relation (244) gives the following associated N' function

$$N' = [n (n + 1)]^{1/2} k_2 (1 + k_2 (L - s))^{-1} . \qquad (246)$$

Substituting the last two relations into (236) and using (237) leads to

$$[n (n + 1)]^{1/2} (2n + 1) k_2{}^2 [1 + k_2 (L - s)]^{-2}$$

$$= \frac{3}{2} \sqrt{\frac{\pi}{2}} (f |\omega|/m' c_1{}^2 \left\{ \int_{-\infty}^{\infty} [1 \right.$$

$$+ k_2 (L - s)]^{-2n} \psi_2 (\omega) d\omega \left. \right\}^{1/2} . \qquad (247)$$

This relation can be satisfied for all s if $n = 2$, but only if $\psi_1 (\omega)$ is a very narrow spectrum centered at some frequency ω_0. As a rough approximation for the case of a realistic spectrum $\psi_1 (\omega)$ we will take ω in the coefficient on the right-hand side of (247) as the mean value, ω_m, defined by

$$\omega_m = \frac{1}{\sigma^2(L)} \int_{-\infty}^{\infty} |\omega| \, \psi_1 (\omega) \, d\omega \ . \tag{248}$$

Then from (247) with $n = 2$

$$\text{\it k}_2 \doteq \left\{ \frac{3}{10} \left(\frac{\pi}{12} \right)^{1/2} \frac{f\sigma \, (L) \, \omega_m}{m' \, C_1{}^2} \right\}^{1/2} \ . \tag{249}$$

The phase function can be obtained from (246) as

$$N \, (s) = \sqrt{6} \, \ell n \, (1 + \text{\it k} \, (L - s) \,). \tag{250}$$

Finally, the variance of $V_\nu{}'$ at any position s is given approximately by

$$\sigma^2 \, (s) \doteq [1 + \text{\it k}_2 \, (L - s)]^{-4} \, \sigma^2 \, (L) \tag{251}$$

for this case, where $\text{\it k}_2$ is given by (249).

The results for the approximate relations for σ (s)
for transverse waves are summarized in Table 3. All
cases assume that the line is sufficiently long that the
variance is attenuated to a negligible value at the
anchor point. Also the approximation is made in each case
that the attenuation rate is based on the mean frequency
ω_m, characterising the spectrum of motion at the top
of the line. The term Γ, defined in Table 3 is a
non-dimensional quantity in which v is either $U \cos \bar{\theta}$
or 1.88σ, depending on the case. The usual conditions
in application, at least for the upper reaches of the
line are for $\Gamma \gg 1$, $\sigma > 2 U \cos \bar{\theta}$. An example plot
of σ vs $(L - s)$ is given in Fig. 19 for the following
conditions:

$$C_d = 1 , \quad m' = 2 m_d,$$

$$D = 1 \text{ inch} , \quad \omega_m = 2\pi/10 \text{ rad/sec} ,$$

$$\sigma (L) = 3 \text{ ft/sec},$$

TABLE 3

APPROXIMATE STATISTICS FOR TRANSVERSE MODES

IN THE EQUILIBRIUM PLANE

	$U \cos \overline{\theta} > 5\ \sigma(s)$	$U \cos \overline{\theta} < 0.5\ \sigma(s)$
Any Γ^*	$v = U \cos \overline{\theta}$	$v = 1.88\ \sigma(s)$
	$\sigma(s) = e^{-\mu(L-s)} \sigma(L)$	$\sigma(s) \doteq [1 + \hbar(L-s)]^{-n}\ \sigma(L)$
$\Gamma \ll 1$	$\mu = \frac{1}{2}\ \Gamma\ \omega_m / C_1$	$\hbar = \frac{1}{2}\ \Gamma\ (L)\ \omega_m / C_1$
		$n = 1$
$\Gamma \gg 1$	$\mu = 0.707\ \Gamma^{1/2}\ \dfrac{\omega_m}{C_1}$	$\hbar = 0.286\ \Gamma(L)^{1/2}\ \dfrac{\omega_m}{C_1}$
		$n = 2$

$^*\ \Gamma \equiv f\ v/m'\ \omega_m$

Fig. 19   Illustration of the attenuation of σ for
transverse waves under the condition of
σ > 2 U cos θ.

where $U \cos \bar{\theta}$ is assumed to be less than $\sigma/2$ at all $(L - s)$. This requires virtually zero value of U for depths greater than 3000 ft if the results of Fig. 20 are to be valid

Case C. Longitudinal Modes: Relations (176C) and (177c) under the condition of zero mean curvature reduce to

$$m \frac{\partial V_{\tau}'}{\partial t} = \frac{\partial T'}{\partial s} \tag{252}$$

$$\frac{\partial \epsilon'}{\partial t} = \frac{\partial V_{\tau}'}{\partial s} \tag{253}$$

which together with relations (178) and (179) govern the longitudinal waves, in the absence of mean curvature. Also for $\bar{T}$ uniform then Y_0 is independent of s.

These relations lead to the following differential equations for V_{τ}' :

$$\frac{\partial}{\partial t} \left\{ \frac{\partial^2 V_{\tau}'}{\partial t^2} - \frac{Y_0}{m} \frac{\partial^2 V_{\tau}'}{\partial s^2} \right\}$$

$$+ \frac{1}{\tau_1} \left\{ \frac{\partial^2 V_{\tau}'}{\partial t^2} - \frac{(Y_0 - K_1)}{m} \frac{\partial^2 V_{\tau}'}{\partial s^2} \right\} = 0. \tag{254}$$

A solution of this relation is

$$V_\tau' = A\, e^{i(\omega t + ks)} \quad , \tag{255}$$

where

$$k^2 = \frac{m\omega^2 \, (1 + i\,\omega\tau_1)}{[(1 + i\,\omega\tau_1)\, Y_0 - K_1]} \quad . \tag{256}$$

For most applications Y_0 greatly exceeds K_1. Under this condition, k has the approximate roots

$$k \doteq \pm\, [(\omega/C_2) - i\,\nu] \quad , \tag{257}$$

where

$$\nu \doteq (\omega/C_2)(K_1/2Y_0)\, \omega\tau_1 / [1 + (\omega\tau_1)^2] \tag{258}$$

and

$$C_2 = \left\{ [Y_0 - K_1\, (1 + (\omega\tau_1)^2)^{-1}]/m \right\}^{1/2}. \tag{259}$$

The attenuation coefficient ν is a small fraction of the wave number for these modes, hence we must definitely allow for reflection of the longitudinal waves at the anchor point so as to assure $V_\tau{}'(o, t) = 0$.

The generalized solutions for an ensemble of possible $V_\tau{}'$ at L is accordingly

$$V_\tau{}'(s,t) = \int_{-\infty}^{\infty} [e^{\nu s}\, e^{i\,\omega(t + s/C_2)}$$

$$- e^{-\nu s}\, e^{i\,\omega(t - s/C_2)}]B^{-1/2}d\,F_3(\omega), \quad (260)$$

where

$$d\,F_3{}^*(\omega)\, dF_3(\omega') = \delta'(\omega - \omega')\,\psi_3(\omega)\,d\omega \qquad (261)$$

and

$$B \equiv 4[\sinh^2(\nu L) + \sin^2(\omega L/C_2)]. \qquad (262)$$

The function $\psi_3(\omega)$ corresponds to the variance spectrum of $V_\tau{}'$ at $s = L$, its integral over all ω being the variance of $V_\tau{}'$ at the vessel. The variance of $V_\tau{}'$ for any position s is given by

$$\overline{(V_T')^2} = \int_{-\infty}^{\infty} \left\{ \frac{\sinh^2 (\nu s) + \sin^2 (\omega s/C_2)}{\sinh^2 (\nu L) + \sin^2 (\omega L/C_2)} \right\} \psi_3 (\omega) \, d\omega . \quad (263)$$

Clearly this vanishes at $s = 0$ as required by the lower boundary condition.

From (252) and (260) it follows that

$$T' = m \int_{-\infty}^{\infty} \frac{i\omega}{(\nu + i\omega/C_2)} \, [\, e^{\nu s} \, e^{i \, \omega(t + s/C_2)}$$

$$+ \, e^{-\nu s} \, e^{i \, \omega(t - s/C_2)}] \; B^{-1/2} \, d \, F_3 (\omega) . \quad (264)$$

Thus the variance of T' using the statistical property (261) is, at any position s,

$$\overline{(T')^2} = m^2 \int_{-\infty}^{\infty} \frac{\omega^2}{[(\omega/C_2)^2 + \nu^2]} \left\{ \frac{\sinh^2 (\nu s) + \cos^2 (\omega s/C_2)}{\sinh^2 (\nu L) + \sin^2 (\omega L/C_2)} \right\}$$

$$\psi_3 (\omega) \, d\omega . \quad (265)$$

Since $\nu \ll \omega/C_2$ and $\nu L \ll 1$ for the normal range of frequencies (even with $L \sim 10^4$ ft), the above relation can be approximated by

$$\overline{(T')^2} = m^2 \int_{-\infty}^{\infty} \frac{C_2{}^2 \cos^2 (\omega s/C_2)}{[\,(\nu L)^2 + \sin^2 (\omega L/C_2)\,]^2} \; \psi_3 (\omega) \; d\omega \; . \qquad (266)$$

As a check, we note that if $\psi_3 (\omega)$ is confined to very small frequencies, implying slow rates of stretching, then the above relation with the use of (259) reduces to approximately

$$\overline{(T')^2} \doteq (Y_0 - K_1)^2 \frac{1}{L^2} \int_{-\infty}^{\infty} \omega^{-2} \psi_3 (\omega) \; d\omega \; , \qquad (267)$$

for any s. Now since $\psi_3 (\omega)$ represents the variance spectrum for $V_\tau{}'$ at $s = L$, then $\omega^{-2} \psi_3 (\omega)$ is the variance spectrum for elongational displacement of the end of the line. Consequently

$$\overline{(\epsilon')^2} = \frac{1}{L^2} \int_{-\infty}^{\infty} \omega^{-2} \psi_3 (\omega) \; d\omega \qquad (268)$$

and

$$\sqrt{\overline{(T')^2}} = (Y_0 - K_1) \sqrt{\overline{(\epsilon')^2}} \; , \qquad (269)$$

as we should expect for slow rate of stretching. Here $(Y_0 - K_1)$ corresponds to the quasi-static spring coefficient, whereas Y_0 is the dynamic coefficient for the given mean tension $\overline{T}$.

On the other hand for a broad-band spectrum, relation (266) indicates the possibility of resonant conditions at frequencies corresponding to zero value of the sin term. These frequencies are $\omega = \pm \omega_n$ where

$$\omega_n = n \pi C_2 / L, \quad n = 1, 2, \ldots \tag{267}$$

For $C_2 \sim 7000$ ft/sec and $L \sim 14,000$ ft this gives $\omega_n \sim n \pi/2$ rad/sec. Near a resonant frequency

$$\sin^2 (\omega L/C_2) \doteq [(\omega - \omega_n) L/C_2]^2$$

$$\tag{268}$$

$$\cos^2 (\omega L/C_2) \doteq 1 .$$

The contribution to the variance of T' at $s = L$ due to a resonant response centered at $\pm \omega_n$ is given approximately by

$$\Delta \overline{(T')^2}_n = 2 m Y_0 \int_0^\infty \frac{\psi_3 (\omega) \, d\omega}{[(\nu L)^2 + (\omega - \omega_n)^2 (L/C_2)^2]} , \tag{269}$$

where it is understood that $\psi_3(-\omega) = \psi_3(\omega)$. Now $\omega_n \tau_1 \sim 2\pi n$ if $\tau_1 \sim 4$ sec as found in Chapter III for nylon line. Hence near the resonance

$$\nu \sim (K_1 / 2 Y_0 C_2 \tau_1)$$

and

$$C_2 \sim (Y_0/m)^{1/2} \ .$$

(270)

Since the resonant response has a very narrow band width (of the order of $K_1 / Y_0 \tau_1$ rad/sec) we can take ψ_3 at the value ω_n in (269) as a suitable approximation. We then obtain upon carrying out the integration

$$\overline{\Delta(T')_n^2} \doteq \frac{4\pi Y_0^3 \tau_1}{K_1 L^2} \ \psi_3(\omega_n) \ ,$$

(271)

for resonant response at $s = L$.

6. Perturbations of a Nylon Mooring Line Relative to a Curved Equilibrium Configuration

It was noted previously that the longitudinal modes and the transverse modes in the plane of the equilibrium configuration are coupled to some extent if the equilibrium configuration possesses curvature. In the following analysis we retain the approximation of uniform $\overline{T}$ and for simplicity we will also suppose the curvature $d\overline{\theta}/ds$ has a uniform value. The pertinent coupled equations are (176a, c), (177a, c), (178), and (179) in which G_ν' is given by (197). In the latter relation we will consider only that case where v is uniform. The above relations then possess solutions of the form

$$(V_\nu', V_\tau', \theta', T', \varepsilon', \xi') = (A_1, A_2, A_3, A_4, A_5, A_6)$$
$$\times e^{i(ks + \omega t)} \qquad (272)$$

provided that the coefficients A_n satisfy the following algebraic relations:

$$(i\omega m' + fv)A_1 = ik\overline{T} A_3 + (d\overline{\theta}/ds) A_4 \cdot \qquad (273a)$$

$$i\omega m A_2 = ik A_4 - \overline{T} (d\overline{\theta}/ds) A_3 , \qquad (273b)$$

$$i\omega (1 + \overline{\varepsilon}) A_3 = ik A_1 + (d\overline{\theta}/ds) A_2 , \qquad (273c)$$

$$i\omega A_5 = ik A_2 - (d\overline{\theta}/ds) A_1 , \qquad (273d)$$

$$(1 + i\omega\tau_1) A_6 = A_5 , \qquad (273e)$$

$$A_4 = Y_0 A_5 - K_1 A_6 . \qquad (273f)$$

The last two of these relations yield

$$A_5 = \frac{(1 + i\omega\tau_1)\, A_4}{[Y_0\,(1 + i\omega\tau_1) - K_1]} \quad . \tag{274}$$

The remaining four relations using (274) will lead to non-trivial values of A_n if k satisfies the characteristic relation obtained by requiring the determinant of the co-efficients in (273) to vanish. This leads to the following possible roots for k^2:

$$
k^2 = \frac{1}{2}\,(k_1{}^2 + k_2{}^2) + (d\bar{\theta}/ds)^2
$$
$$
\pm \left\{ \frac{1}{4}\,(k_1{}^2 - k_2{}^2)^2 + (k_0{}^2 + k_1{}^2 + k_2{}^2 + k_1{}^2\,k_2{}^2/k_0{}^2) \right.
$$
$$
\left. \times (d\bar{\theta}/ds)^2 \right\}^{1/2} , \tag{275}
$$

where

$$k_0{}^2 \equiv m\,\omega^2\,(1 + \bar{\epsilon})/T, \tag{276a}$$

$$k_1{}^2 \equiv \omega(1 + \bar{\epsilon})(\omega\,m' - i\,fv)/T , \tag{276b}$$

$$k_2{}^2 \equiv \frac{m\,\omega^2\,(1 + i\,\omega\,\tau_1)}{[Y_0\,(1 + i\,\omega\,\tau_1) - K_1]} \quad . \tag{276c}$$

As a special case we note that for $d\overline{\theta}/ds = 0$ the two roots for k^2 are simply $k_1{}^2$ and $k_2{}^2$ which correspond to relations (203) and (256) in the previous analyses for transverse and longitudinal wave modes respect''ve.y. For typical conditions:

$$|k_0{}^2/k_1{}^2| \sim 1 ,$$

$$|k_2{}^2/k_1{}^2| \sim 10^{-2} , \qquad (277)$$

$$|(d\overline{\theta}/ds)/k_1{}^2| \sim 10^{-2} .$$

Accordingly, in the case of relatively small curvature (275) reduces approximately to

$$k^2 \doteq \begin{cases} k_1{}^2 \\ \\ k_2{}^2 - \left(\dfrac{m'}{m} - i\,\dfrac{fv}{m\omega}\right)^{-1} \left(\dfrac{d\overline{\theta}}{ds}\right)^2 , \end{cases} \qquad (278)$$

for transverse and longitudinal modes respectively. One effect of the curvature is to increase the rate of attenuation of the longitudinal waves. For the case of $fv/m\omega \gg 1$

$$k_2 \doteq \pm \left(\omega/C_2 - i\,\nu'\right) , \qquad (279a)$$

where ν' is the attenuation coefficient

$$\nu' = \nu + \frac{mC_2}{fv}\left(\frac{d\bar{\theta}}{ds}\right)^2 , \tag{279b}$$

ν being given by (258) and C_2 by (259). The longitudinal waves can lose energy not only by hysteresis, but also by net transfer to the transverse waves by coupling due to the curvature. The transferred energy is ultimately dissipated by form drag.

Returning to relations (273) we find the following approximate relations among A_1, A_2, A_3, A_4, making use of (277):

For transverse modes:

$$A_2' \doteq -(i/k_1)(d\bar{\theta}/ds)\, A_1',$$

$$A_3' \doteq (k_1/\omega\,(1 + \bar{\epsilon}))\, A_1', \tag{280}$$

$$A_4' \doteq -(i\, m\, \omega/k_1{}^2)(1 + k_1{}^2/k_0{}^2)(d\bar{\theta}/ds)\, A_1' ;$$

For longitudinal modes:

$$A_1{}'' = -(i/k_2)(k_0{}^2/k_1{}^2)(d\overline{\theta}/ds)\ A_2{}'' ,$$

$$A_3{}' = -(i\ m\ddot{w}/k_1{}^2)(1 + k_1{}^2/k_0{}^2)(d\overline{\theta}/ds)\ A_2{}'' , \qquad (281)$$

$$A_4{}'' = (m\omega/k_2)\ A_2{}'' .$$

Note that if $d\overline{\theta}/ds = 0$ then $A_2{}'$, $A_4{}'$, and $A_1{}''$, $A_3{}''$ would vanish.

The general solutions for $V_\nu{}'$ and $V_\tau{}'$, which are accurate to first order in $d\overline{\theta}/ds$ and yield essentially zero values at $s = 0$, are

$$V_\nu{}'(s,t) = \int_{-\infty}^{\infty} e^{i\omega t + ik_1(s - L)}\ dF_1(\omega)$$

$$+ \int_{-\infty}^{\infty} \frac{d\overline{\theta}}{ds} \frac{k_0{}^2}{k_2 k_1{}^2} \left[e^{-ik_1 s} - \cos k_2 s \right] e^{i\omega t} dF_3(\omega)$$

$$(282a)$$

and

$$V_\tau{}'(s,t) = \int_{-\infty}^{\infty} \sin k_2 s\ e^{i\omega t}\ dF_3(\omega)$$

$$- \int_{-\infty}^{\infty} \frac{i}{k_1} \frac{d\overline{\theta}}{ds} e^{i\omega t + ik_1(s - L)}\ dF_1(\omega) , \qquad (282b)$$

where $dF_1(\omega)$ and $dF_3(\omega)$ remain to be determined in terms of the representations of V_ν' and V_τ' at $s = L$. In the above relations

$$k_1 = \varkappa - i\mu ,$$

$$\tag{283}$$

$$k_2 = \omega/C_2 - i\nu' ,$$

where $\mu L \gg 1$.

Let

$$V_\nu'(L,t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dH_1(\omega) \tag{284a}$$

$$V_\tau'(L,t) = \int_{-\infty}^{\infty} e^{i\omega t}\, dH_3(\omega) \tag{284b}$$

where

$$\overline{dH_1^*(\omega)\, dH_1'(\omega)} = \delta'(\omega - \omega')\psi_1(\omega)\, d\omega ,$$

$$\tag{285}$$

$$\overline{dH_3^*(\omega)\, dH_3(\omega')} = \delta'(\omega - \omega')\psi_3(\omega)\, d\omega .$$

In (285) $\psi_1(\omega)$ and $\psi_3(\omega)$ are the variance spectra for V_ν' and V_τ' respectively at $s = L$. Moreover we will suppose that V_ν' and V_τ' at $s = L$ possess

a cross-spectrum $\psi_{13}(\omega)$ such that

$$\overline{dH_1{}^*(\omega)\ dH_3(\omega')} = \delta'(\omega - \omega')\ \psi_{13}(\omega)\ d\omega. \qquad (286)$$

If we now equate (282a, b) at $s = L$ with (284 a, b) respectively we can solve for $dF_1(\omega)$ and $dF_3(\omega)$ in terms of $dH_1(\omega)$ and $dH_3(\omega)$. Inserting the resulting relations in (282a, b) and neglecting terms of order $(d\overline{\theta}/ds)^2$ yields

$$V_\nu{}'(s,t) = \int_{-\infty}^{\infty} e^{ik_1(s-L)}\ e^{i\omega t}\ dH_1(\omega)$$

$$+ \frac{d\overline{\theta}}{ds} \int_{-\infty}^{\infty} \frac{k_0{}^2}{k_2 k_1{}^2} \left\{ (e^{-ik_1 s} - \cos k_2 s) \right.$$

$$\left. - e^{ik_1(s-L)}(e^{-ik_1 L} - \cos k_2 L) \right\} \frac{e^{i\omega t}}{\sin k_2 L}\ dH_3(\omega) \quad (287)$$

and

$$V_\tau{}'(s,t) = \int_{-\infty}^{\infty} \frac{\sin k_2 s}{\sin k_2 L}\ e^{i\omega t}\ dH_3(\omega)$$

$$- \frac{d\overline{\theta}}{ds} \int_{-\infty}^{\infty} \frac{i}{k_1} \left\{ e^{ik_1(s-L)} - \frac{\sin k_2 s}{\sin k_2 L} \right\} e^{i\omega t}\ dH_1(\omega). \quad (288)$$

The solution for T' to the same order of approximation can be shown to be given by

$$T'(s,t) = \int_{-\infty}^{\infty} \frac{m_\omega}{ik_2} \frac{\cos k_2 s}{\sin k_2 L} e^{i\omega t} dH_3(\omega)$$

$$+ \frac{d\bar{\theta}}{ds} \int_{-\infty}^{\infty} \frac{m_\omega}{k_1^2} \left[\frac{k_1}{k_2} \frac{\cos k_2 s}{\sin k_2 L} \right.$$

$$\left. - i(1 + k_1^2/k_0^2) e^{ik_1(s-L)} \right] e^{i\omega t} dH_1(\omega) .$$

$$(289)$$

The primary effects of the curvature are: (a) to cause the transverse oscillations to reach a greater depth and (b) to produce a somewhat greater variance of T' at the upper end of the line compared to that which would exist without curvature of the line. At intermediate depths along the line, the contribution to the variance of V_ν' by the first integral of (287) becomes negligible (as seen in the previous section) and the second integral gives a residual variance given approximately by

$$(V_\nu{}')^2 \quad \sim \quad \left(\frac{d\bar\theta}{ds}\right)^2 \int\limits_{-\infty}^{\infty} \left| \frac{k_0 \lambda'}{k_2 k_1 \lambda'} \right|^2 \psi_3(\omega) \, d\omega \qquad\qquad (290)$$

at mid-depths.

It may be remarked in closing that the effect of curvature is of more significance in the case of steel cables where the net weight in water produces a relatively greater value of curvature than for nylon line, other conditions being comparable. In fact for nylon line, the simpler theory given in the preceding section in which curvature is ignored is probably adequate for most conditions.

Further application of the methods discussed herein would be profitable, particularly in respect to the use of the generalized upper end conditions discussed in Chapter II. Unfortunately this is well beyond the scope of the present study, but it is hoped that the present study will serve as a useful guide in future research on the problem of mooring line dynamics.

REFERENCES

Freiberger, W. F., Editor (1960). International Dictionary
of Applied Mathematics, D. Van Nostrand Co., Princeton,
1173 pp.

Fofonoff, N. P., (1965). A technique for analyzing buoy
system motion, Geo-Marine Technology, July, 1965,
pp V-222-225.

Fofonoff, N. P., (1966). Oscillation modes of a deep-sea
mooring, Geo-Marine Technology, Oct., 1966, pp
V-327-331.

Freeman, J. C., Jr., (1951). The solution of non-linear
meteorologoical problems by the method of characteristics,
Compendium of Meteorology, Boston, Amer. Met. Soc.,
pp 421-433.

Gover, S. C., (1955). Observations on ship motions at
sea. Proceedings of the First Conference on Ships
and Waves. Council on Wave Research and the
Society of Naval Architects and Marine Engineers.
pp 160-184.

Lamb, H., Sir, (1945). Hydrodynamics, Dover Publication,
Sixth Edition, New York, 738 pp.

Meixner, J., (1965). Thermodynamic theory of relaxational
phenomena, Non-Equilibrium Thermodynamics, Variational
Techniques and Stability, Edited by R. J. Donnelly,
P. Hermand and I. Prigogine, University of Chicago
Press, Chicago, pp 72-89.

O'Brien, J. T. and B. J. Muga, (1963). Sea tests on
a swing-moored aircraft carrier, U. S. Naval
Civil Engineering Laboratory, Technical Report,
R. 251, 79 pp.

O'Brien, J. J. and R. O. Reid, (1967). The non-linear
response of a two-layer baroclinic ocean to a
stationary, axially symmetric hurricane: Part I.
Upwelling induced by momentum transfer, J. of
Atm. Sci., 24 (2):197-207.

Paquette, R. G. and B. E. Henderson, (1965). Dynamics of simple deep-sea buoy moorings, General Motors Defense Research Lab, Tech. Report 65-79, 167 pp.

Polachek, H., T. S. Walton, R. Mejia, and C. Dawson, (1963). Transient motion of an elastic cable immersed in a fluid, *Mathematics of Computation*, pp 60-63.

Reid, R. O. and B. W. Wilson, (1963). Boundary flow along a circular cylinder, *Jour. of Hydraulics Division*, Amer. Soc. Civil Eng., 89 (HY3), pp 21-40.

Schlichting, H., (1960). Boundary Layer Theory, McGraw-Hill Co., New York, 647 pp.

St. Denis, Manley and W. J. Pierson, (1955). On some recent developments in the theory of ship motions, Proceedings of the First Conference on Ships and Waves, Council on Wave Research and the Society of Naval Architects and Marine Engineers, pp 160-184.

Walton, T. S. and H. Polachek, (1960). alculation of transient motion of submerged cables, *Mathematical Tables and Aids to Computation*, 14, pp 27-45.

Wiener, Norbert (1950), Extrapolation, interpolation and smoothing of stationary time series, John Wiley and Sons, Inc., New York, 163 pp.

Whicker, L. F. (1958). Theoretical Analysis of the effect of ship motion on mooring cables in deep water, U. S. Navy David Taylor Model Basin, Report 1221, 24 pp.

Wilson, B. W. (1964). Characteristics of anchor cables in uniform ocean currents, Bulletin Nos. 11 and 12, Permanent Internatl. Assoc. Navig. Congr., 69 pp.

Wilson, B. W. (1965). Characteristics of deep-sea anchor cables in strong currents, Bulletin No. 17 and 18, Permanent Internatl. Assoc. Navig. Congr., 65 pp.

Wilson, B. W. (1967). Elastic characteristics of moorings, Jour. of Waterways and Harbors Division, Amer. Soc. Civil Engr., pp 27-56.

Zajac, E. E., (1957). Dynamics and kinematics of the laying and recovery of submarine cable, Bell System Tech. Jour., 36 (5): pp 1129-1207.

APPENDIX A.   EQUATIONS OF MOTION OF VESSEL

We wish to establish relations connecting the
motion of a moored vessel with the potential motion of
the same vessel in the same seaway if it were not moored
and not underway.

First, consider the equations of motion for the
unmoored vessel. Let λ_j (j = 1, 2, 3) denote the three
Cartesian components of the position of the center of
the mass of the vessel (relative to a fixed equilibrium
position). Let α_j (j = 1, 2, 3) represent the com-
ponents of angular displacement about the mutually
orthogonal, principal axes* of the vessel. Assuming
the vessel is symmetrical about its mid-section, one
principal axis is normal to the plane of symmetry and
the other two lie in the plane of symmetry. Under
equilibrium conditions it will be assumed that the
plane of symmetry is vertical and that the longitudinal
principal axis is horizontal. It will be assumed more-
over, that the motions are sufficiently small such that
the non-linear coupling effects of the translational
and rotational components of motion can be ignored.

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* The principal axes are those for which the moment
of inertia matrix can be represented in diagonalized form.

Following St. Denis and Pierson (1955), the linearized
equations of motion for an unmoored vessel are taken
as follows (j = 1, 2,):

$$M_j \; d^2\lambda_j/dt^2 + N_j \; d\lambda_j/dt + K_j \; \lambda_j = F_j' \; , \qquad (A-1)$$

$$I_j \; d^2\alpha_j/dt^2 + N_j' \; d\alpha_j/dt + K_j' \; \alpha_j = H_j' \; , \qquad (A-2)$$

where the M_j are the effective masses for surge, sway,
and heave (including the effective added mass of water),
the N_j are damping coefficients associated with
viscous drag and/or radiation of ship induced waves,
the K_j are restoring force coefficients (force per
unit displacement) and the F_j' are the components of
exciting force due to wave motion and/or steady motion
of the water relative to the vessel. Similarly the
I_j are the effective moments of inertia about the
principal axes of the vessel, the N_j' are damping
coefficients, the K_j' are restoring coefficients
(torque per radian) and the H_j' are components of
exciting torque due to waves. It will be understood
specifically that λ_1, λ_2, λ_3 are respectively the
translational displacements in surge, sway, and heave.

moreover, α_1, α_2, α_3 are respectively the rotational displacements in roll, pitch, and yaw. Accordingly K_1, K_2, and K_3' are zero; the only restoring forces being those for heave, roll, and pitch.

We can rewrite (A-1) and (A-2) in the form

$$d^2\lambda_j/dt^2 + 2\beta_j \, d\lambda_j/dt + \sigma^2_j \, \lambda_j = f_j' \, , \qquad \text{(A-3)}$$

$$d^2\alpha_j/dt^2 + 2\beta_j' \, d\alpha_j/dt + \sigma_j'^2 \, \lambda_j = h_j' \, , \qquad \text{(A-4)}$$

where

$$\beta_j \equiv N_j/2M_j \, ,$$

$$\beta_j' \equiv N_j'/2I_j \, ,$$

$$\sigma_j \equiv \sqrt{K_j/M_j} \, ,$$

$$\sigma_j' \equiv \sqrt{K_j'/I_j} \, , \qquad \text{(A-5)}$$

$$f_j' \equiv F_j'/M_j \, ,$$

and

$$h_j' \equiv H_j'/I_j \, .$$

The four sets of coefficients β_j, β_j', σ_j and σ_j' all have units of frequency; the β terms govern the damping rate for free motion. The natural frequencies

(radians/second), for the case where $\beta_3 \ll \sigma_3$, $\beta_1' \ll \sigma_1'$ and $\beta_2' \ll \sigma_2'$, are given approximately by σ_3, σ_1' and σ_2' (for heave, roll, and pitch respectively). Again note that σ_1, σ_2 and σ_3' are zero. It will be presumed that all six damping rates and the three natural frequencies for the unmoored vessel are known.

For the <u>moored</u> vessel we must include the effect of the tensile force for each mooring line. Let $\hat{T}_k(t)$ denote the tensile force exerted by mooring line k ($k = 1, 2, \ldots$) at its point of attachment to the vessel. Let $\hat{p}_k$ represent the vector separation between the vessel's center of mass the the kth attachment point (see Fig. 2). Recalling the convention for $\hat{T}$, we note that the force applied to the vessel by line k is $-\hat{T}_k$. Likewise line k exerts a torque given by $-\hat{p}_k \times \hat{T}_k$. Let T_{kj} represent the jth Cartesian component of the vector T_k and let p_{kj} be the jth component of the separation vector p_k, the components being referred to the same coordinate system as that for λ_j and α_j. Then $\hat{p}_k \times \hat{T}_k$ has the components J_{kj} given by:

$$J_{k_1} = (p_{k_2} T_{k_3} - p_{k_3} T_{k_2})$$

$$J_{k_2} = (p_{k_3} T_{k_1} - p_{k_1} T_{k_3}) \qquad \text{(A-6)}$$

$$J_{k_3} = (p_{k_1} T_{k_2} - p_{k_2} T_{k_1}) \; .$$

If we now let R_j denote the translational displacements of the center of mass for the moored vessel and ψ_j the angular displacements then the equations of motion for the moored vessel take the form

$$d^2 R_j/dt^2 + 2\beta_j \, dR_j/dt + \sigma_j^2 \, R_j = f_j - \sum_K T_{kj}/M_j \qquad \text{(A-7)}$$

$$d^2 \psi_j/dt^2 + 2\beta_j{}' \, d\psi_j/dt + \sigma_j{}'^2 \, \psi_j = h_j - \sum_K J_{kj}/I_j \qquad \text{(A-8)}$$

where J_{kj} is given by (A-6) and the terms f_j and h_j are the counterparts of $f_j{}'$ and $h_j{}'$ for the moored vessel.

Suppose (A-3), (A-4), (A-7), and (A-8) are applied for the _same vessel_ and for the _same sea state_. Moreover suppose for the unmoored condition, that the vessel is not underway. It is reasonable to suppose then that the mean values of the exciting terms f_j and h_j are zero.

This is not true for the moored situation. However, it will be assumed that the departures of f_j and of h_j from their mean values are the same as f_j' and h_j' ; i.e.,

$$f_j - \overline{f}_j = f_j'$$

and   (A-9)

$$h_j - \overline{h}_j = h_j'.$$

We expect that the mean values of the time derivatives of R_j and of $\dot{\psi}_j$ will vanish and hence

$$\overline{f}_j = \sigma_j^2 \, \overline{R}_j + \sum_k \overline{T}_{kj}/M_j$$

$$\overline{h}_j = \sigma_j'^2 \, \overline{\psi}_j + \sum_k \overline{J}_{kj}/I_j \; .$$

(A-10)

Consequently from (A-3), (A-4), (A-7), (A-8), (A-9), and (A-10) we obtain the following relations for the anomalies of ship motion induced by the mooring line:

$$(d^2/dt^2 + 2\beta_j \, d/dt + \sigma_j^2) \, R_j'$$
$$= -\sum_k T_{kj}'/M_j \; , \qquad (A-11)$$

$$(d^2/dt^2 + 2\beta_j' \, d/tdt + \sigma_j'^2) \, \psi_j'$$

$$= -\sum_k J_{kj}'/I_j \qquad \text{(A-12)}$$

where

$$R_j' \equiv R_j - \overline{R}_j - \lambda_j, \quad T_{kj}' \equiv T_{kj} - \overline{T}_{kj},$$

$$\text{(A-13)}$$

$$\psi_j' \equiv \psi_j - \overline{\psi}_j - \alpha_j, \quad J_{kj}' \equiv J_{kj} - \overline{J}_{kj}.$$

It is understood that λ_j and α_j are prescribed functions of t; these are the potential displacements which would exist in the absence of the mooring lines for the same sea state.

To these relations we must add a kinematical relation governing the orientation of the vectors $\hat{p}_k$. For a rigid vessel the separation distances $|\hat{p}_k|$ are constant, however, the orientation is governed by the vector angular velocity of the vessel, $\hat{\Omega}$, according to the relation

$$d\hat{p}_k/dt = \hat{\Omega} \times \hat{p}_k . \qquad \text{(A-14)}$$

The Cartesian components of $\hat{\Omega}$ are given by $d\psi_j/dt$. Hence (A-14) can be expressed in the following component form:

$$(d/dt)\ p_{k2} = (d\psi_2/dt)\ p_{k3} - (d\psi_3/dt)\ p_{k2} \quad,$$

$$(d/dt)\ p_{k2} = (d\psi_3/dt)\ p_{k1} - (d\psi_1/dt)\ p_{k3} \quad,$$

and

$$(d/dt)\ p_{k3} = (d\psi_1/dt)\ p_{k2} - (d\psi_2/dt)\ p_{k1} \quad. \qquad (A\text{-}15)$$

The restriction has already been imposed that the motions
of the vessel are small. Consequently we may approximate
relations (A-15) by taking the mean values of the terms
p_{kj} on the right hand sides and obtain by integration,
using (A-13):

$$p_{k1}{}' = (\psi_2{}' + \alpha_2)\ p_{k3}{}^o - (\psi_3{}' + \alpha_3)\ p_{k2}{}^o \quad,$$

$$p_{k2}{}' = (\psi_3{}' + \alpha_3)\ p_{k1}{}^o - (\psi_1{}' + \alpha_1)\ p_{k3}{}^o \quad, \qquad (A\text{-}16)$$

$$p_{k3}{}' = (\psi_1{}' + \alpha_1)\ p_{k2}{}^o - (\psi_2{}' + \alpha_2)\ p_{k1}{}^o \quad,$$

where

$$p_{kj}{}' = p_{kj} - p_{kj}{}^o \qquad (A\text{-}17)$$

and $p_{kj}{}^o$ denotes the mean values of p_{kj}.

Finally to the above relations we added the following geometrical relation for each line

$$\hat{r}_k \;=\; \hat{R} + \hat{p}_k + \hat{C}_k \tag{A-18}$$

where $\hat{r}_k$ is the radius vector from the anchor point of line k to the point of attachment on the vessel, while $\hat{C}_k$ is a constant vector representing the radius vector from the anchor point for line k to the fixed reference point for the displacement vector $\hat{R}$. The above relation can also be written in the form

$$\hat{r}_k \;=\; \hat{r}_k{}^o + \hat{R}' + \hat{p}_k' + \hat{\lambda} \tag{A-19}$$

where $\hat{r}_k{}^o$ is the mean value of $\hat{r}_k$.

We note in passing that the vector $\hat{p}_k$ can be written in the form

$$\hat{p}_k \;=\; \hat{p}_k{}^o + (\hat{\psi}' + \hat{\alpha}) \times \hat{p}_k{}^o \; . \tag{A-20}$$

This is consistent with (A-16) and (A-17).

APPENDIX B. DERIVATION OF THE CHARACTERISTIC
FORM OF THE MOORING LINE EQUATIONS

The equations governing the local time rates of change of the dependent variables V_ν, V_μ, V_τ, θ, Φ, ϵ, ξ, and ϵ_0 are given by (109), (110), (111), (93), (94), (95), (70), and (71), respectively. These equations also contain the dependent variable T, however, we have the diagnostic equation (69) relating T to ϵ, ξ and ϵ_0. We wish to recast these equations in the form

$$\sum_{n=1}^{8} B_n \frac{dQ_n}{dt} = P, \tag{B-1}$$

along the path

$$\frac{ds}{dt} = C . \tag{B-2}$$

Here the Q_n signify the eight prognostic variables and the quantities B_n, P and C are functions of these variables. In general there will be eight possible values of the speed C (not necessarily all distinct) and eight associated relations of form (B-1). All

of the admissible values of C will be real if the
system of differential equations is hyperbolic, which
is true in the present case.

First of all, we obtain from (69)

$$\frac{\partial T}{\partial s} = Y \left(\frac{\partial \epsilon}{\partial s} - \frac{\partial \epsilon_0}{\partial s}\right) - K_1 \frac{\partial \xi}{\partial s} ,$$
(B-3)

where

$$Y \equiv K e^{b(\epsilon - \epsilon_0)} .$$
(B-4)

This relation can be employed in (111) so as to eliminate
any dependency on derivatives of T.

In order to put the system of equations in the
form (B-1) with condition (B-2), we form a quasi-linear
combination of the original set in the following sense.
Multiply (109), (110), and (111) by b_1, b_2, and b_3
respectively;, multiply (93), (94), and (95) by b_4, b_5,
and b_6 respectively; multiply (70) and (71) by b_7 and
b_8, respectively; and add the resulting equations. The
combination is quasi-linear in the sense that the co-
efficients b_1 through b_8 do not involve derivatives
of the dependent variables. After collecting terms, the
result is:

$$[b_1 \ m' \ \frac{\partial V_\nu}{\partial t} - b_4 \ \frac{\partial V_\nu}{\partial s}] \ + \ [b_2 \ m' \ \frac{\partial V_\mu}{\partial t} - b_5 \ \frac{\partial V_\mu}{\partial s}]$$

$$+ \ [b_3 \ m \ \frac{\partial V_\tau}{\partial t} - b_6 \ \frac{\partial V_\tau}{\partial t}] \ + \ [b_1 \ m' \ V_\tau - b_3 \ X_2 + (1 + \epsilon)b_4]\frac{\partial \theta}{\partial t}$$

$$+ \ (- b_1 \ T - b_4 \ V_\tau + b_6 \ V_\nu) \ \frac{\partial \theta}{\partial s}]$$

$$+ \ [\ (- b_1 \ m' \ V_\mu \cos \theta + b_2 \ m' \ X_1 - b_3 \ X_3 \sin \theta$$

$$+ \ b_5 \ (1 + \epsilon) \sin \theta) \frac{\partial \Phi}{\partial t} + (- b_2 \ T \sin \theta + b_4 \ V_\mu \sin \theta$$

$$- \ b_5 \ X_1 + b_6 \ V_\mu \sin \theta) \ \frac{\partial \Phi}{\partial s}]$$

$$+ \ [b_6 \ \frac{\partial \epsilon}{\partial t} - b_3 \ Y \ \frac{\partial \epsilon}{\partial s}] + [b_7 \ \frac{\partial \xi}{\partial t} + b_3 \ K_1 \ \frac{\partial \xi}{\partial s}]$$

$$+ \ [b_8 \ \frac{\partial \epsilon_0}{\partial t} + b_3 \ Y \ \frac{\partial \epsilon_0}{\partial s}] \ = \ b_1 \ (w \sin \theta + G_\nu)$$

$$+ \ b_2 \ G_\mu + b_3 \ (- w \cos \theta + G_\tau) + b_7 \ (\epsilon - \epsilon_0 - \xi)/\tau_1$$

$$+ \ b_8 \ S \ (T, \ \epsilon_0) \quad . \tag{B-5}$$

In the above relation

$$X_1 \equiv V_\nu \cos\theta + V_\tau \sin\theta \ ,$$

$$X_2 \equiv m' V_\nu - m_a U_\nu \qquad\qquad\qquad \text{(B-6)}$$

$$X_3 \equiv m' V_\mu - m_a U_\mu$$

$$S(T, \epsilon_0) \equiv \begin{cases} (F(T) - \epsilon_0)/\tau_0, & \text{if } \epsilon_0 < F(T) \\ 0 & , \text{ if } \epsilon_0 \geq F(T) \end{cases} \qquad \text{(B-7)}$$

and

$$m' = m + m_a. \qquad\qquad\qquad\qquad \text{(B-8)}$$

The coefficients b_1 through b_8 are to be selected such that

$$b_4 = - C m' b_1 \ , \qquad\qquad\qquad \text{(B-9)}$$

$$b_5 = - C m' b_2 \ , \qquad\qquad\qquad \text{(B-10)}$$

$$b_6 = - C m \ b_3 \ , \qquad\qquad\qquad \text{(B-11)}$$

$$b_3 \, Y = - C \, b_6 \ , \qquad\qquad\qquad \text{(B-12)}$$

$$b_3 \, K_1 = C \, b_7 \ , \qquad\qquad\qquad \text{(B-13)}$$

$$b_3 \, Y = C \, b_8 \ , \qquad\qquad\qquad \text{(B-14)}$$

$$(- T\, b_1 - V_\tau\, b_4 + V_\nu\, b_6) \; =$$

$$C\, (m'\, V_\tau\, b_1 - X_2\, b_3 + (1 + \epsilon)\, b_4), \qquad (B\text{-}15)$$

and

$$(- T \sin \theta\, b_2 + V_\mu \cos \theta\, b_4 - X_1\, b_5 + V_\mu \sin \theta\, b_6)$$

$$= C\, (- m'\, V_\mu \cos \theta\, b_1 + m'\, X_1\, b_2 - X_3 \sin \theta\, b_3$$

$$+ (1 + \epsilon) \sin \theta\, b_5). \qquad (B\text{-}16)$$

The latter relations assure that each of the dependent variables, Q_n, in (B-5) enters in the form

$$(\frac{\partial Q_n}{\partial t} \; + \; C\, \frac{\partial Q_n}{\partial s}) \; ,$$

which is the same as dQ_n/dt along the line $ds/dt = C$.

Using (B-9), relation (B-15) can be simplified to the form

$$(- T\, b_1 + V_\nu\, b_6) \; = \; C\, (- X_2\, b_3 + (1 + \epsilon)\, b_4). \quad (B\text{-}17)$$

Moreover, using (B-9) and (B-10), relation (B-16) can
be reduced to the form

$$(- T\, b_2 + V_\mu\, b_6) = C\, (- X_3\, b_3 + (1 + \epsilon)\, b_5), \quad (B-18)$$

provided that $\sin\theta$ does not vanish.

Since the eight homogeneous relations (B-9, through
(B-14) plus (B-17) and (B-18) contain eight parameters,
b_n, plus an eigenvalue C, it follows that at least
one of the b_n for given C is arbitrary. The eigenvalues
C can be evaluated as the roots of the eighth degree
polynominal in C found from the requirement that the
determinant of the coefficients in the simultaneous
system of equations for b_n vanishes. This determinant
in the present case contains many zeros and can be shown to
lead to the following eighth degree polynominal:

$$C^2\, (Y - C^2 m)\, (C^2 - (1 + \epsilon) - T)^2 = 0. \quad (B-19)$$

Therefore, C^2 can have the possible roots

$$0, \quad Y/m \quad \text{or} \quad T/m'\, (1 + \epsilon),$$

the last having a multiplicity of two.

For the case $C^2 = 0$ we find that the relations for b_n are satisfied if $b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 0$ for arbitrary b_7 and b_8. Thus for one of the two roots $C = 0$, take $b_7 = 1$, $b_8 = 0$ and for the other take $b_7 = 0$, $b_8 = 1$. This implies that we merely recover relations (70) and (71) in their original form, as might have been anticipated, since they involve only derivatives of ξ and ϵ_0 respectively.

For the case C^2 $T/m'(1 + \epsilon)$, one possible set of b_n are found to be $b_2 = b_3 = b_5 = b_6 = b_7 = b_8 = 0$ with b_1 arbitrary and b_4 given by

$$b_4 = -C\,m'\,b_1 \;.$$

Thus if we take $b_1 = 1$, then

$$b_4 = \mp [\, m'\,T/(1 + \epsilon)]^{1/2}$$

for   (B-20)

$$C = \pm\, [T/m'(1 + \epsilon)]^{1/2}.$$

A second possible set of b_n for the case $C^2 = T/m'(1 + \epsilon)$ is found to be $b_1 = b_3 = b_4 = b_6 = b_7 = b_8 = 0$ with b_2 arbitrary and b_5 given by

$$b_5 \;=\; -\,C\,m'\,b_2 .$$

Thus with $b_2 = 1$ we get

$$b_5 \;=\; \mp\,[\,m'\,T/(1 + \epsilon)]^{1/2}$$

for

$$C \;=\; \pm\,(T/m'\,(1 + \epsilon))^{1/2}.$$

(B-21)

Finally for the case $C^2 = Y/m$ we get from (B-9) to (B-14) with (B-17) and (B-18) the following relations for the b_n :

$$b_1 \;=\; -C\,m_a\,(V_\nu - U_\nu)/Z ,$$

$$b_2 \;=\; -C\,m_a\,(V_\mu - U_\mu)/Z ,$$

$$b_3 \;=\; 1 ,$$

$$b_4 \;=\; Y\,m'\,m_a\,(V_\nu - U_\nu)/mZ ,$$

$$b_5 \;=\; Y\,m'\,m_a\,(V_\mu - U_\mu)/mZ ,$$

$$b_6 \;=\; -Y/C,$$

$$b_7 \;=\; K_1/C ,$$

$$b_8 \;=\; Y/C ,$$

(B-22)

where

$$C = \pm (Y/m)^{1/2}$$

and

$$Z \equiv \frac{m'}{m} (1 + \epsilon) Y - T. \tag{B-23}$$

Since the spring coefficient, Y, is generally much greater than T, the quantity Z is positive (even when T is at or near its ultimate value).

The relations of form (B-1) follow immediately from the above results by substituion of the b_n into (B-5) for each particular C. In the case $C = \pm (Y/m)^{1/2}$, the terms involving $d\epsilon/dt$, $d\xi/dt$, and $d\epsilon_0/dt$ can be recollected using the relation

$$\frac{dT}{dt} = Y \frac{d}{dt} (\epsilon - \epsilon_0) - K_1 \frac{d\xi}{dt}. \tag{B-24}$$

The resulting characteristic form of the mooring line equations, making use of definitions (B-6), are those presented in Section 4 of Chapter V.

APPENDIX C. TRANSFORMATION OF THE CHARACTERISTIC
EQUATIONS TO CARTESIAN FORM

Using (88) gives the following relations between the natural and Cartesian velocity components

$$V_\mu = -V_x \sin \Phi + V_y \cos \Phi \, ,$$

$$V_\nu = V_x \cos \theta \cos \Phi + V_y \cos \theta \sin \Phi - V_z \sin \theta \, ,$$

$$\text{(C-1)}$$

$$V_\tau = V_x \sin \theta \cos \Phi + V_y \sin \theta \sin \Phi + V_z \cos \theta \, .$$

These are employed in (115), (116), and (117) to replace dV_μ/dt, dV_ν/dt, and dV_τ/dt by the time derivatives of V_x, V_y, V_z, θ, and Φ.

Now from the definitions given in (132) we find

$$\frac{d\alpha}{dt} = \cos \Phi \cos \theta \frac{d\theta}{dt} - \sin \Phi \sin \theta \frac{d\Phi}{dt} \, ,$$

$$\text{(C-2)}$$

$$\frac{d\beta}{dt} = \sin \Phi \cos \theta \frac{d\theta}{dt} + \cos \Phi \sin \theta \frac{d\Phi}{dt} \, .$$

These relations are employed to eliminate $d\theta/dt$ and $d\Phi/dt$ by appropriate combination of the transformed

versions of (115), (116), and (117). Finally using (132) the coefficients involving $\sin \theta$, $\cos \theta$, $\sin \Phi$, $\cos \phi$ can be put in terms of α, β, γ. The resulting relations are those given by (137) to (139). The last of these was given in abbreviated form involving a residual term $\overset{+}{\delta}$. The complete form for (139) is as follows:

$$\frac{dV_x}{dt} + \beta \frac{dV_y}{dt} + \gamma \frac{dV_z}{dt} \mp (mY)^{-1/2} \frac{dT}{dt}$$

$$= (G_\tau - w\gamma)/m - (Y/m)^{1/2} [K_1 (\epsilon - \epsilon_0 - \xi)/Y\tau_1$$

$$+ S(T, \epsilon_0)] + \left\{ \mp (Y/m)^{1/2} \frac{m_a}{mZ} [(G_\nu + w \sqrt{\alpha^2 + \beta^2}) (V_\nu - U_\nu) \right.$$

$$+ G_\mu (V_\mu - U_\mu)] - \frac{m_a T}{xZ} [(V_x - U_x) \frac{d\alpha}{dt} + (V_y - U_y) \frac{d\beta}{dt}$$

$$+ (V_z - U_z) \frac{d\gamma}{dt}] + (Y/m)^{1/2} \frac{m_a m'}{mZ} (\hat{V}_n - \hat{U}_n) \cdot \frac{d\hat{V}}{dt} \right\} \quad (C-3)$$

along $ds/dt = \pm (Y/m)^{1/2}$.

The lengthy set of terms in braces represents the residual $\overset{+}{\delta}$. In this relation

$$Z \equiv \frac{m'}{m} (1 + \epsilon) Y - T. \tag{C-4}$$

Fortunately under normal conditions the residual $\delta^{\pm}$ is of very minor consequence and can be neglected for practical purposes. To justify this we note, first of all, that Z is of the order of magnitude of Y. Thus the terms involving G_ν and G_μ are of the order of magnitude

$$\frac{(V_n - U_n)}{C''} \cdot \frac{G_n}{m} ,$$

where $C'' = (Y/m)^{1/2}$. Under normal conditions we would expect that $(V_n - U_n)$ is of the order of magnitude of 10 ft/sec or less. However, C'' is of the order of 10,000 ft/sec. Thus the above terms are negligible compared with $(G_\tau - w\gamma)/m$.

The terms involving $d\alpha/dt$, $d\beta/dt$, and $d\gamma/dt$ are of the order

$$(C'/C'')^2 \ (V_n - U_n) \frac{d\theta}{dt} ,$$

where C' is the signal speed for transverse waves. The ratio $(C'/C'')^2$ is of the order $1/100$. If $(V_n - U_n) \ d\theta/dt$ is of the order $\alpha dV_x/dt$, then the above terms can be neglected.

Finally, the terms involving $d\hat{V}/dt$ in the residual $\delta^{\pm}$ are of the order

$$\frac{(V_n - U_n)}{C''} \frac{dV_n}{dt} .$$

The ratio $(V_n - U_n)/C''$ is of the order 10^{-3}, hence these terms are negligible compared with $\alpha \, d \, V_x/dt$, $\beta \, d \, V_y/dt$, and $\gamma \, d \, V_z/dt$.

It will be noted furthermore that all of the terms in the residual $\delta^{\pm}$ are proportional to m_a. These terms would therefore not exist in the absence of the fluid. The primary influence of the fluid is clearly on the transverse waves and not on the longitudinal waves.

APPENDIX D. MATRIX OF COEFFICIENTS FOR
THE NUMERICAL PREDICTION RELATIONS

The coefficients $M_{n,m}$ and P_n of relations (154) are given in the following pages. These are arranged in order according to n.

$\underline{n = 1}$

$$M_{11} = \left\{ 2 - [\ \alpha^2(j,k+1) + \alpha^2(A_1)] \right\}/B_1 ,$$

$$M_{12} = -\left[\alpha(j,k+1)\ \beta(j,k+1) + \alpha(A_1)\ \beta(A_1) \right]/B_1 ,$$

$$M_{13} = -\left[\alpha(j,k+1)\ \gamma(j,k+1) + \alpha(A_1)\ \gamma(A_1) \right]\Big/B_1 ,$$

$$M_{14} = -1 ,$$

$$M_{15} = M_{16} = M_{17} = M_{18} = 0 ,$$

$$P_1 = \left\{ 2 \sum_{m=1}^{8} M_{1,m}\ Q_m(A_1) + \left[G_{nx}(j,k+1) + G_{nx}(A_1) \right. \right.$$

$$\left. \left. + w(\alpha(j,k+1)\ \gamma(j,k+1) + \alpha(A_1)\ \gamma(A_1)) \right]\ \delta t_1/m' \right\}/B_1 ,$$

$$B_1 = \left\{ [\ (1 + \epsilon(j,k+1))\ T(j,k+1)/m']^{1/2} \right.$$

$$\left. + [(1 + \epsilon(A_1))\ T(A_1)/m']^{1/2} \right\} .$$

$$C_1' = \frac{1}{2} \left\{ \left[\frac{T(j,k+1)}{m'(1 + \epsilon(j,k+1))} \right]^{1/2} + \left[\frac{T(A_1)}{m'(1 + \epsilon(A_1))} \right]^{1/2} \right\} .$$

$\underline{n = 2}$

$$M_{21} = \left\{ 2 - [\alpha^2 (j,k+1) + \alpha^2(A_2)] \right\}/B_2 \ ,$$

$$M_{22} = - \left[\alpha(j,k+1) \ \beta(j,k+1) + \alpha(A_2) \ \beta(A_2) \right]/B_2 \ ,$$

$$M_{23} = - \left[\alpha(j,k+1) \ \gamma(j,k+1) + \alpha(A_2) \ \gamma(A_2) \right]/B_2 \ ,$$

$$M_{24} = 1 \ ,$$

$$M_{25} = M_{26} = M_{27} = M_{28} = 0 \ ,$$

$$P_2 = \left\{ 2 \sum_{m=1}^{8} M_{2,m} \ Q_m \ (A_2) + [G_{nx} \ (j,k+1) + G_{nx} \ (A_2) \right.$$

$$\left. + w(\alpha(j,k+1) \ \gamma(j,k+1) + \alpha(A_2) \ \gamma(A_2)] \ \delta t_2/m' \right\}/B_2 \ ,$$

$$B_2 = \left\{ [(1 + \epsilon(j,k+1)) \ T(j,k+1)/m']^{1/2} \right.$$

$$\left. + [(1 + \epsilon(A_2) \ T(A_2)/m']^{1/2} \right\} \ ,$$

$$C_2' = \frac{1}{2} \ \left\{ \left[\frac{T(j,k+1)}{m'(1 + \epsilon(j,k+1))} \right]^{1/2} + \left[\frac{T(A_2)}{m')1 + \epsilon(A_2))} \right]^{1/2} \right\} \ .$$

<u>n = 3</u>

$$M_{31} = M_{12} \ , \quad (A_3 = A_1) \ ,$$

$$M_{32} = \left\{ 2 - \left[\beta^2(j,k+1) + \beta^2(A_1) \right] \right\}/B_1 \ ,$$

$$M_{33} = - \left[\beta(j,k+1) \, \gamma(j,k+1) + \beta(A_1) \, \gamma(A_1) \right]/B_1 \ ,$$

$$M_{34} = 0 \ ,$$

$$M_{35} = -1 \ ,$$

$$M_{36} = M_{37} = M_{38} = 0 \ ,$$

$$P_3 = \left\{ 2 \sum_{m=1}^{8} M_{3,m} \, Q_m(A_1) + \left[G_{ny}(j,k+1) + G_{ny}(A_1) \right.\right.$$

$$\left.\left. + \ w \, (\beta(j,k+1) \, \gamma(j,k+1) + \beta(A_1) \, \gamma(A_1) \right] \delta t_1 /m' \right\}/B_1 \ ,$$

and

$$C_3' = C_1' \ .$$

$\underline{n = 4}$

$$M_{41} = M_{22} \qquad (A_4 = A_2)$$

$$M_{42} = \left\{ 2 - \left[\beta^2 \ (j,k+1) + \beta^2 \ (A_2) \right] \right\}/B_2$$

$$M_{43} = - \left[\beta(j,k+1) \ \gamma(j,k+1) + \beta(A_2) \ \gamma(A_2) \right]/B_2$$

$$M_{44} = 0$$

$$M_{45} = 1$$

$$M_{46} = M_{47} = M_{48} = 0$$

$$P_4 = \left\{ 2 \sum_{m=1}^{8} M_{4m} \ Q_m \ (A_2) + \left[G_{ny} \ (j,k+1) + G_{ny} \ (A_2) \right. \right.$$

$$\left. \left. + w \ (\beta(j,k+1) \ \gamma(j,k+1) + \beta(A_2) \ \gamma(A_2)) \right] \ \delta t_2/m' \right\}/B_2$$

and

$$C_4' = C_2'$$

$\underline{n\ =\ 5}$

$$M_{51} = [\alpha(j,k+1) + \alpha(A_5)]/B_3$$

$$M_{52} = [\beta(j,k+1) + \beta(A_5)]/B_3$$

$$M_{53} = [\gamma(j,k+1) + \gamma(A_5)]/B_3$$

$$M_{54} = M_{55} = 0$$

$$M_{56} = -1$$

$$M_{57} = M_{58} = 0$$

$$P_5 = \left\{ 2 \sum_{m=1}^{8} M_{5m}\ Q_m\ (A_5) + \left[G_\tau(j,k+1) + G_\tau(A_5) \right. \right.$$

$$\left. - w\ (\gamma(j,k+1) + \gamma(A_5)) \right]\ \delta t_5 /m$$

$$\left. + C_5''\ \left[\ \widetilde{F}(j,k+1) + \widetilde{F}(A_5) \right]\ \delta t_5 \right\}\ /B_3$$

$$B_3 = 1/m\ C_5''\ ,\ \ \widetilde{F} \equiv \frac{K_1}{y\tau_1}\ (\epsilon - \epsilon_0 - \xi) + S(T,\ \epsilon_0)$$

$$C_5'' = \frac{1}{2}\ \left[\ (Y(j,k+1)/m)^{1/2} + (Y(A_5)/m)^{1/2}\ \right]$$

$\underline{n = 6}$

$M_{61} = [\alpha(j,k+1) + \alpha(A_6)]/B_4$

$M_{62} = [\beta(j,k+1) + \beta(A_6)]/B_4$

$M_{63} = [\gamma(j,k+1) + \gamma(A_6)]/B_4$

$M_{64} = M_{65} = 0$

$M_{66} = 1$

$M_{67} = M_{68} = 0$

$P_6 = \left\{ 2 \sum_{m=1}^{8} M_{6m} Q_m (A_6) + \left[G_\tau(j,k+1) + G_\tau (A_6) \right.\right.$

$\left. - w(\gamma(j,k+1) + \gamma(A_6)) \right] \delta t_6/m$

$\left. + C_6'' \left[\widetilde{F}(j,k+1) + \widetilde{F}(A_6) \right] \delta t_a \right\}/B_4$

$B_4 = 1/m \ C_6''$

$C_6'' = \frac{1}{2} \left[(Y(j,k+1)/m)^{1/2} + (Y(A_6)/m)^{1/2} \right]$

$\underline{n \ = \ 7}$

$$M_{71} = M_{72} = M_{73} = M_{74} = M_{75} = M_{76} = 0$$

$$M_{77} = 1, \quad M_{78} = 0$$

$$P_7 = \left\{ \left[(1/b) \ln[Y(j,k+1) \ Y(j,k)] - \xi(j,k) \right] \Delta t/2\tau_1 \right.$$

$$\left. + \bar{\varepsilon}(j,k) \right\} \ / \ (1 + \Delta t/2\tau_1)$$

$\underline{n \ = \ 8}$

$$M_{81} = M_{82} = M_{83} = M_{84} = M_{85} = M_{86} = 0$$

$$M_{87} = 0, \quad M_{88} = 1$$

$$P_8 = \left\{ \varepsilon_0 \ (j,k) + F(T(j,k)) \ \Delta t/\tau_0 \right\}/(1 + \Delta t/\tau_0) \ ,$$

$$\text{If} \quad \varepsilon_0(j,k) < F(T(j,k))$$

$$P_8 = \varepsilon_0(j,k), \quad \text{If} \quad \varepsilon_0(j,k) > F(T(j,k)) \ .$$

LIST OF SYMBOLS

A	Cross-sectional area of mooring line at s, t; also used as an arbitrary constant
A_n	End point of characteristic line (Fig. 16)
a_n	Value of $s/\Delta s$ for point A_n; Also used to denote normal acceleration
$\hat{B}$	Archemedian force on line
B_n	End point of characteristic line (Fig. 17)
$B_n^{\pm}$	Coefficients in characteristic equations
b	Rate of change of the dynamic spring coefficient with tension, see Eq (55)
b_n	Interpolated relative time for point B_n
C	Signal speed (C' for transverse modes, C'' for longitudinal modes)
C_1, C_2	Transverse and longitudinal signal speeds corresponding to conditions of the mean state of the line
C_n	See Eq (144)
c_a	Added mass coefficient
c_d	Drag coefficient (form drag)
D	Effective diameter of the line at s, t
D_o	Nominal diameter of the line in its original unstrained state
D	Mean rate of energy dissipation by form drag per unit material length of line

$dF_n(\omega)$ $dH_n(\omega)$ $dQ_n(\omega)$	Stochastic, complex differentials employed in Fourier, Stieltjes representations of stochastic variables
E	Total mechanical energy per unit material length of line
E_e	Elastic energy per unit material length of line
E_k	Effective kinetic energy per unit material length of line
E_p	Potential energy per unit material length of line
$\hat{F}$	Total fluid force on line
$F(T)$	Saturation function giving ϵ_o vs T for slow unidirectional straining
$\hat{f}$	Fluid force per unit material length of line
$\hat{f}_b, \hat{f}_d, \hat{f}_i, \hat{f}_s$	Contribution to $\hat{f}$ due respectively to static buoyancy, form drag, acceleration of line relative to fluid and lateral thrust due to vortex shedding
$f(\alpha)$	A scalar function of α
f	Coefficient in relation for $\hat{G}$ (Chap. IV), see Eq (173)
$\hat{G}$	That part of residual fluid force $\hat{R}$ not dependent upon $\partial\hat{r}/\partial t$, see Eq (105) or (106)
G_ν, G_μ, G_τ	Natural components of G
G_{nx}, G_{ny}	x and y components of the normal part of $\hat{G}$
G'_ν, G'_μ	Anomalies of G_ν and G_μ, see Eq (175)
$G(\lambda)$	Kernal function in an integral representation for T, see Eq (73)

g	Acceleration due to gravity
H	Hysteresis for a closed cycle (energy loss per cycle per unit material length of line)
H_i	Hysteresis for an idealized process
H_m	Maximum value of H
$h(\omega)$	Transfer function related to $G(\lambda)$
I_j	Moments of inertia of vessel about its principal axes for roll, pitch, and yaw ($j = 1,2,3$ respectively)
$\hat{i}$	Unit vector in the x-direction a fixed Cartesian frame of reference
i	$\sqrt{-1}$; also used to denote an integer
$\hat{J}_k$	The negative of the torque exerted on the surface vessel by mooring line k
$\hat{J}_k'$	Anomaly of $\hat{J}_k$ from its mean value $(\hat{J}_k^o)$
J_{kj}	Cartesian components of $\hat{J}_k$ ($j = 1,2,3$)
J_e	The value 0 if $a_n < 0$, N if $a_n > N$
J_n	The integer truncation of the number a_n
$\hat{j}$	Unit vector in the y-direction for a fixed Cartesian frame of reference
j	An integer
K	Dynamic spring coefficient at zero tension, see Eq (55)
K_o, K_1	Constants with units of load per unit strain characterizing the elastic properties of the line
$K*$	Apparent dynamic spring coefficient for cyclic loading at frequency ω

$\hat{k}$	Unit vector in the z-direction (upwards) for a fixed Cartesian frame of reference
k	An integer; also denotes a complex wave number (Chapter IV).
k_1, k_2	Constants in the representation of the damping characteristics for transverse oscillations of the line
L	Original, unstrained length of li.. ($s = L$ at point of attachment to ve..el)
M	Total mass of line (directly a function of s)
M_j	The effective inertial mass of the surface vessel for surge, sway, and heave ($j = 1, 2, 3$ respectively)
$M_{n,m}$	Matrix of coefficients in the equations for Q_m, see Eq (154) and Appendix D.
m	Mass of line per unit material coordinate
m_a	Effective added mass of water per unit material coordinate
m_d	Mass of water displaced by line per unit material coordinate
m'	$m + m_a$
N	Frictional coefficient ($= K\tau$) for line; also an integer representing the maximum number of intervals of s in the numerical grid
0	Notation for anchor point of mooring line
P	Denotes a material point on the line; also denotes period ($= 2\pi/w$).
$P^{\pm}$	Forcing terms in the characteristic equations

P_n — Forcing term in Eq (154), see Appendix D

$\hat{p}_k$ — Separation vector between center of mass of vessel and point of attachment of mooring line k (See Fig. 2)

$\hat{p}_k^o$ — Mean value of vector $\hat{p}_k$

p_{kj} — Cartesian components of $\hat{p}_k$ ($j = 1, 2, 3$)

$Q(j, k)$ — Denotes any dependent variable of the mooring line at $s = j\Delta s$, $t = l\Delta t$

Q_n — The set of dependent variables defined by Eq (152), $n = 1, 2 \ldots 8$

$\hat{q}_i{}'$ — Cartesian components of the velocity of the center of mass of the surface vessel

$\hat{R}$ — Residual fluid force per unit material coordinate, see Eq (18)

$\hat{R}'$ — Anomaly of the translational displacement of the surface vessel

R_j' — Components of $\hat{R}'$ in surge, sway and heave ($j = 1, 2, 3$ respectively)

$\hat{r}$ — Position vector of a material point on the line with reference to the anchor point

$\hat{r}_k$ — Position vector at point of attachment of line k on the surface vessel

$\hat{r}_k{}^o$ — Mean value of $\hat{r}_k$

$S(T, \epsilon_0)$ — A function defined by Eq (119c)

s — Material coordinate, representing the arc length between O and P in the original relaxed state of the line (a function of M alone)

$\hat{T}$ — Vector tension in line at s, t ($\hat{T} = T\hat{\tau}$)

$\hat{T}_0$	Value of $\hat{T}$ at anchor point
$\hat{T}_k$	Value of $\hat{T}$ at point of attachment of line k to surface vessel
$\hat{T}_k{}^o$	Mean value of $\hat{T}_k$
$\hat{T}_{kj}$	Cartesian components of $\hat{T}_k$ (j - 1,2,3)
$\hat{T}_k'$	Anomaly of $\hat{T}_k$ from its mean value $(\hat{T}_k{}^o)$
T	Scalar tension
T_d	Dynamic tension for given ϵ and ϵ_0
T_s	Saturation value of T for given ϵ_0
T_u	Ultimate strength of line
$\overline{T}$	Mean value of T at s
T'	Perturbation of T from its mean value at s
t	time
$\hat{U}$	Velocity of fluid (in absence of line) at s,t
$\hat{U}_n$	That part of $\hat{U}$ normal to the line at s,t
U_ν, U_μ, U_τ	Natural components of $\hat{U}$
U_x, U_y, U_z	Cartesian components of $\hat{U}$
$\dot{U}_x, \dot{U}_y, \dot{U}_z$	Cartesian components of $\partial\hat{U}/\partial t$
$\hat{V}$	Velocity of line at s,t $(\partial\hat{r}/\partial t)$
$\hat{V}_n$	$\hat{V} - \hat{\tau} V_\tau$
V_ν, V_μ, V_τ	Natural components of $\hat{V}$ at s,t
V_x, V_y, V_z	Cartesian components of $\hat{V}$ at s,t
V_ν', V_μ', V_τ'	Components of $\hat{V}$ in a natural coordinate system defined by the _mean_ configuration of the line

v	Effective mean value of the relative normal velocity, defined by Eq (193)
w	Net weight of line in water per unit material coordinate
x	Horizontal Cartesian coordinate in the plane of the equilibrium configuration of the line (for coplanar conditions)
Y	The general spring coefficient $dT/d\epsilon$, see Eq (117c) or (121)
Y_o	Value of Y corresponding to $\overline{T}$ and $\overline{\xi}$
y	Horizontal Cartesian coordinate normal to the Coplanar equilibrium configuration of the line
Z	See Eq (126)
z	Vertical Cartesian coordinate

$\hat{\alpha}$	A vector denoting the potential rotational displacement of the surface vessel
α_j	Cartesian components of $\hat{\alpha}$ in roll, pitch, and yaw ($j = 1,2,3$ respectively)
α	A dummy variable to denote $\epsilon - \epsilon_0$; also used to denote the x-component of $\hat{\tau}$
$\beta_j, \beta_j{}'$	Damping factors pertinent to the six components of motion of the surface vessel (β_j for translation, $\beta_j{}'$ for rotation, $j = 1,2,3$)
β	The y-component of $\hat{\tau}$
β'	The anomaly of β from its mean value
Γ	See Eq (107); also used in a different sense in Table 3
γ	The z-component of $\hat{\tau}$

Δs	Increment of s
ΔT	Amplitude of T under cyclic loading
Δt	Uniform time step for numerical computation
δ	An arbitrary phase angle
$\delta'(\alpha)$	Dirac delta function
$\delta^{\pm}$	See page 196
δt	Partial time step given by Eq (153c)
ϵ	Longitudinal strain in the line at s,t (defined by $\partial\sigma/\partial s -1$)
ϵ_0	Permanent strain (or passive analastic strain)
ϵ_1	A partial strain referred to in the discussion of the Maxwell model
ϵ'	Perturbation of ϵ from its mean value at s
ζ	See Eq (188)
η	Ratio of lateral thrust on line to drag force
θ	Zenith angle of $\hat{\tau}$, see Fig 15
$\bar{\theta}$	Mean value of θ at s
θ'	Perturbation of θ from its mean at s
$\varkappa$	Real part of the complex wave number k for transverse wave modes
$\hat{\lambda}$	Potential translational displacement of the surface vessel
λ_j	Cartesian component of $\hat{\lambda}$ in surge, sway, and heave ($j=1,2,3$ respectively)
λ	See Eq (107)

$\hat{\mu}$ — Unit vector in the natural coordinate system normal to $\hat{\tau}$ and $\hat{k}$ at s,t

$\hat{\mu}_0$ — Mean orientation of $\hat{\mu}$ at s

$\hat{\mu}'$ — Departure of $\hat{\mu}$ from $\hat{\mu}_0$

μ — Imaginary part of the complex wave number κ for transverse wave modes

$\hat{\nu}$ — Unit vector in the natural coordinate system normal to $\hat{\tau}$ and $\hat{\mu}$ at s,t

$\hat{\nu}_0$ — Mean orientation of $\hat{\nu}$

$\hat{\nu}'$ — Departure of $\hat{\nu}$ from $\hat{\nu}_0$

ν — Imaginary part of the complex wave number for longitudinal waves (in the absence of curvature)

ν' — Modified value of ν for longitudinal waves in the presence of curvature of the line

ξ — Active anelastic strain at s,t (its meaning is clarified by reference to Chap III, Sec. 2)

$\bar{\xi}$ — Mean value of ξ at s

ξ' — Perturbation of ξ from its mean at s

π — 3.14159 . . .

ρ — Average density of sea water (about 2 slugs per ft^3)

σ — Actual arc length of line between O and P in the general strained state at time t

σ_j, σ_j' — Nominal natural frequencies for an unmoored vessel (σ_1, σ_2, σ_3' being zero)

$\sigma_1{}^2, \sigma_2{}^2$ — Variance of V_ν' and V_μ' respectively at s

$\hat{\tau}$ — Unit vector tangent to line at s, t

$\hat{\tau}_0$ — Mean oreintation of $\hat{\tau}$ at s

$\hat{\tau}'$ — Departure of $\hat{\tau}$ from $\hat{\tau}_0$ at s

τ_0, τ_1	Characteristic relaxation times for the line
Φ	Azimuth of $\hat{\tau}$, see Fig. 15
$\hat{\psi}'$	Anomaly of the rotational displacement of the surface vessel
ψ_j'	Cartesian components of $\hat{\psi}'$ in roll, pitch, and yaw ($j = 1, 2, 3$ respectively)
$\psi_1(\omega)$	Variance spectrum for V_ν' at $s = L$
$\psi_2(\omega)$	Variance spectrum for V_μ' at $s = L$
$\psi_3(\omega)$	Variance spectrum for V_τ' at $s = L$
Ω_i'	Cartesian components of the anomaly of angular velocity of the surface vessel ($i = 1, 2, 3$ respectively for roll, pitch, and yaw)
ω	Radian frequency
ω_m	Spectral mean value of ω, see Eq (248)
ω_s	Strouhal frequency

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13. ABSTRACT

Equations of motion for the three-dimensional aspects of mooring line dynamics are formulated and dealt with in a coordinate system which facilitates analysis of the transverse and longitudinal modes of motion. A visco-elastic model for the mooring line material is adopted which allows for hysteresis effects and non-linear load-strain processes. A numerical procedure for solution based upon the method of characteristics is discussed in some detail. The linearized equations for three-dimensional small perturbations relative to a coplanar equilibrium configuration are formulated and applied to the problem in which the motion is stipulated in a stochastic sense at the upper end of the line. More general conditions at the upper end are discussed but not applied in the present study.